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This guide contains an outline of topics to be included in individual subject areas in secondary school mathematics and some specific suggestions for teaching them. Areas covered include--(1) fundamentals of mathematics included in seventh and eighth grades and general mathematics in the high school, (2) algebra concepts for courses one and two, (3) geometry, and (4) advanced mathematics. The guide was written with the following purposes in mind--(1) to assist local groups to have a basis on which to plan a mathematics course of study, (2) to give individual teachers an overview of a particular course or several courses, and (3) to provide specific suggestions for teaching such topics. (RP)

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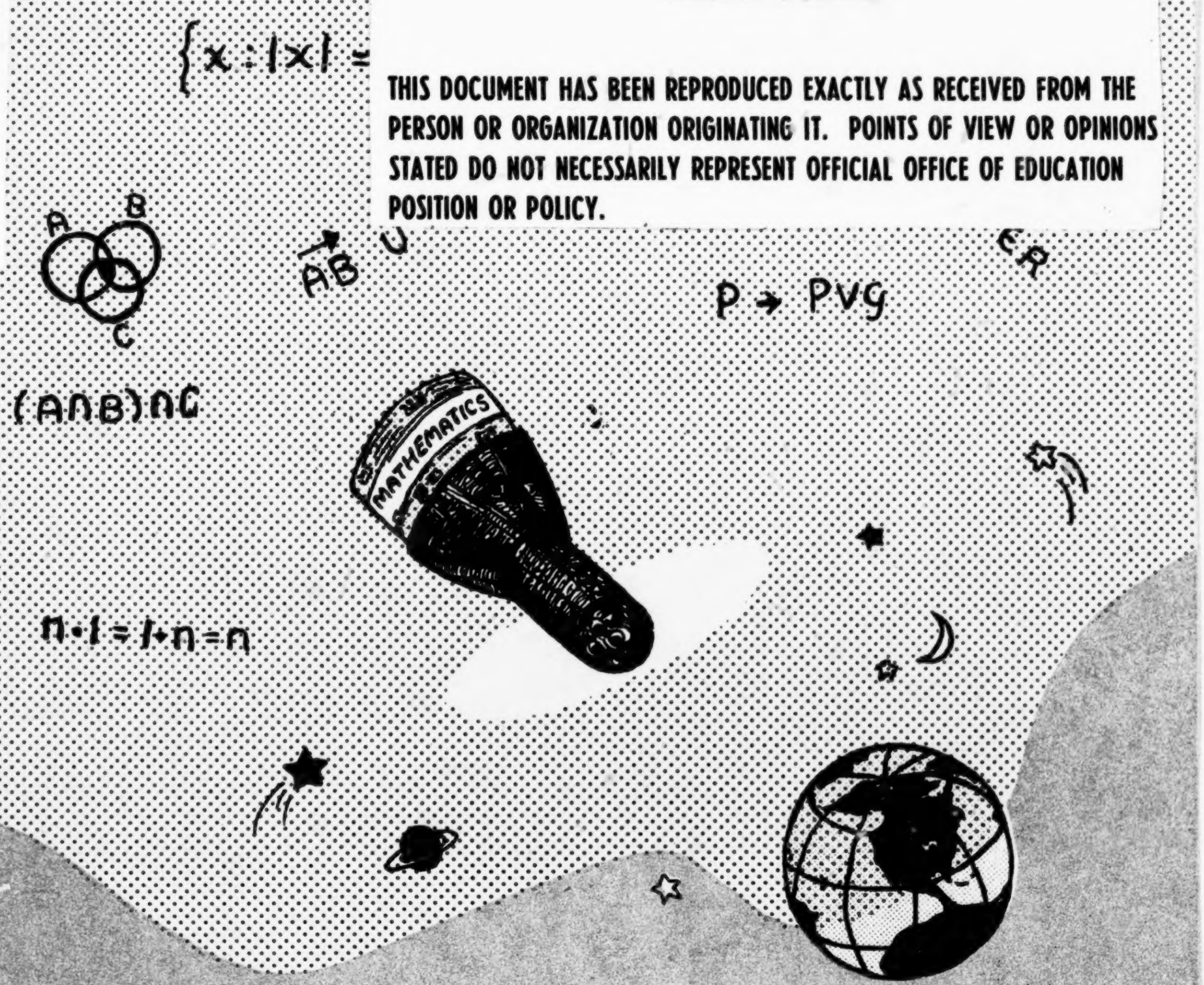


Guidelines for

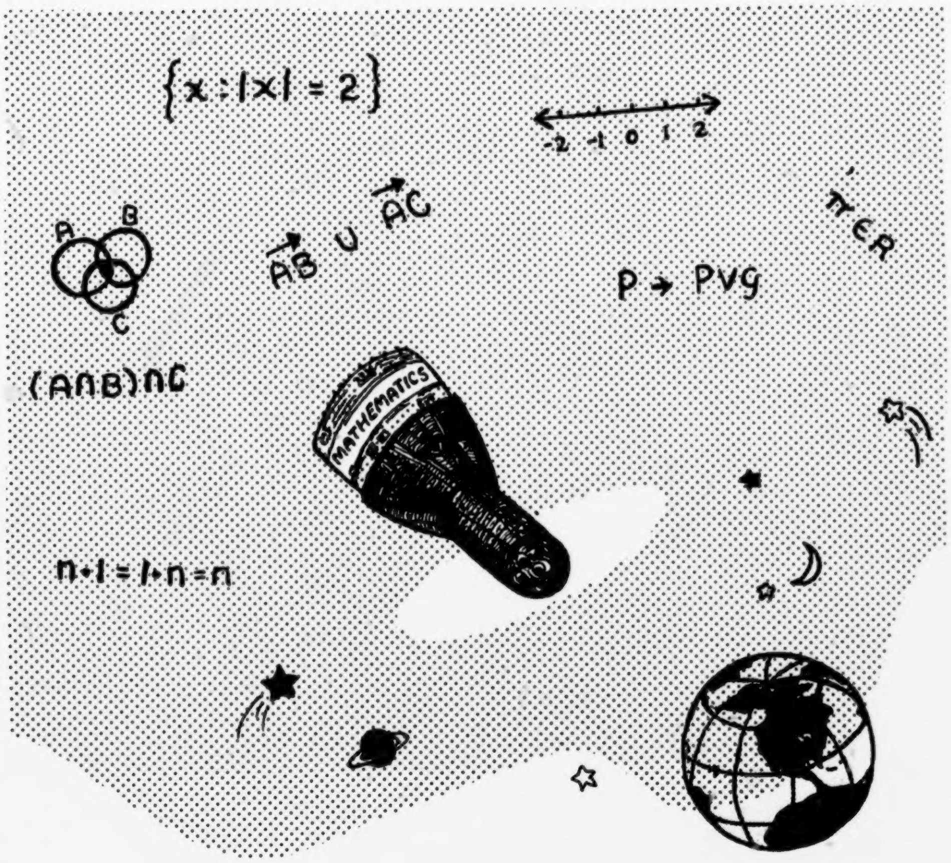
MATHEMATICS IN THE SECONDARY SCHOOL

1965

SE 004 576

STATE DEPARTMENT OF EDUCATION
Columbia, S. C.

JESSE T. ANDERSON
State Superintendent of Education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MATHEMATICS
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Guidelines for
MATHEMATICS IN THE SECONDARY SCHOOL

Published by

STATE DEPARTMENT OF EDUCATION
JESSE T. ANDERSON, State Superintendent
Columbia, S. C.
1965

Permission to Reprint

Permission to reprint *A Guide, Mathematics in Florida Secondary Schools* has been granted by the State Department of Education, Tallahassee, Florida, Thomas D. Bailey, Superintendent.

The South Carolina State Department of Education is indebted to the Florida State Department of Education and the authors of *A Guide, Mathematics in Florida Secondary Schools*. We are grateful for the privilege of making this material available to the teachers of mathematics in South Carolina schools.

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Foreword

MATHEMATICS IN our present world is continuing to make an increased contribution to our culture as well as being an important element of scientific education. If mathematics is to continue to contribute to the advancement of civilization, then it is essential that it be well selected and well taught. Throughout the nation, there is evidence that mathematics education in the secondary schools is being strengthened.

The South Carolina State Department of Education recognizes the need for a set of guidelines for secondary school mathematics in order to assist schools in improving their mathematics programs. An excellent curriculum guide published by the Florida State Department of Education was found that has much to offer the secondary school program in South Carolina. Rather than wait to develop such a guide from resources in South Carolina, permission to reprint the Florida guide was requested and granted.

The South Carolina State Department of Education offers this publication hoping that it will assist local school personnel to improve the teaching of mathematics.

JESSE T. ANDERSON
State Superintendent of Education

Preface

Many South Carolina schools are making considerable progress in improving the quality of mathematics education in the secondary schools. The spirit of change began gathering momentum with the production of curriculum materials by foundation-sponsored study groups as early as 1958. Inherent in these improved programs are teachers whose qualifications are up-dated and the implementation of up-dated curriculum materials that have grown out of experimental studies.

Many schools have moved into improved programs and those that are contemplating such a move are looking for help and for information on what concepts should be taught. To this end the South Carolina State Department of Education offers this publication which is essentially a reproduction of a publication published by the Florida State Department of Education.

The content of this publication is intended only as a suggested guide. It has proven very valuable to the schools of Florida, and on the recommendation of Mr. Daniel H. Sandel and a statewide advisory committee, we offer it to the secondary schools of our state. We hope this booklet will assist local groups to have a basis on which to plan a mathematics course of study, give individual teachers an overview of a particular course or several courses, and provide specific suggestions for teaching some topics.

J. Carlisle Holler, Director
Division of Instruction

Acknowledgments

A NUMBER OF teachers, administrators, college professors, and State Department of Education personnel have worked steadily for months on the development of this new guide to teaching mathematics. In addition, classroom teachers, students, and even parents have been involved in experimenting with and evaluating prior to publication the effectiveness of the material contained in this bulletin. To identify by name all participants in this project is impractical, although the importance of each contribution is recognized and genuinely appreciated.

Acknowledgement should be extended individually, however, to the members of the state-wide curriculum guide committee who assumed the greatest share of the responsibility for developing this new curriculum bulletin. Dr. Robert Kalin, Associate Professor of Mathematics Education, Florida State University, served as chairman of the Committee. Membership of the Committee comprised, in addition to the Chairman, Mrs. Jessie Bailey, Teacher of Mathematics, Blountstown High School; Mr. S. Roger Bailey, Teacher of Science, Clearwater High School; Miss Elizabeth Cauley, Mathematics Instructor, Pensacola Junior College; Miss Lucille Craver, Counselor, King High School, Tampa; Mr. Louis M. Edwards, Curriculum Assistant in Mathematics, Orange County Schools; Mr. George F. Evans, Supervisor of Mathematics, Broward County Schools; Dr. Kenneth P. Kidd, Professor of Education, University of Florida; Mrs. Jeannette Pomeroy, Teacher of Mathematics, Clearlake Junior High School, Cocoa; Mrs. Agnes Rickey, Supervisor, Mathematics, Dade County Schools; and Mr. Leroy B. Smith, Coordinator of Mathematics, Duval County Schools. To all these people, who worked diligently and long to write and present for teacher use this new aid to instruction, is due sincere gratitude for a superior job.

The unexpected and untimely deaths of Mr. Wayland Phillips, who was serving as Mathematics Consultant with the State De-

partment of Education, and Dr. Robert C. Yates, who was Head of the Department of Mathematics at the University of South Florida, cut short their participation in this project, but both men had already made substantial contributions and had left their indelible marks on the work. It is expected that this guide will stand as another monument to their many contributions to the teaching of mathematics in Florida and to their positive influence on the public-school curriculum.

Appreciation is also due to the Florida Council of Teachers of Mathematics for its interest in and support of the project.

Grateful acknowledgment is extended to the staff members of the State Department of Education who assisted the Committee with the development and publication of this guide. Dr. Fred W. Turner, Director, Division of Instructional Services, and Mr. Dexter Magers, Supervisor, Mathematics Education, deserve special recognition for the leadership, encouragement, and strong professional support they gave the project. Special recognition is also due Dr. Joseph W. Crenshaw, Assistant Director, General Education and Curriculum, Division of Instructional Services, who reviewed the material, made helpful editorial contributions, and assumed major responsibility for publication of the guide.

We are further indebted to Mr. J. K. Chapman, Mr. Howard Jay Friedman, and Mr. R. W. Sinclair for professional and technical advice and assistance with illustration, lay-out, and preparation of the guide for publication and distribution.

Recognition

APPRECIATION IS expressed to the members of the South Carolina Mathematics Advisory Committee for their evaluation and suggestions pertinent to these materials. This committee is composed of:

Mr. Cecil Clark; Beaufort High School; Beaufort, South Carolina

Mrs. Lucille Free; Denmark-Olar High School; Denmark, South Carolina

Miss Lucille Huggin; Spartanburg High; Spartanburg, South Carolina

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Cover design by Gerry Thomas

Introduction

THIS GUIDE contains an outline of topics to be included in individual subject areas in secondary school mathematics and some specific suggestions for teaching them. Areas covered include:

1. Fundamentals of mathematics embraced in seventh, and eighth grades and general mathematics in the high school;
2. Algebra concepts for courses one and two;
3. Geometry;
4. Advanced mathematics.

Use of the Guide

The outline and suggestions do not constitute a rigorous prescription imposed by the State Department of Education or the Advisory Committee. The purpose of the guide is to: (1) give schools in South Carolina a measuring device for analyzing their mathematics program; (2) assist local groups to have a basis on which to plan a mathematics course of study; (3) give individual teachers an overview of a particular course or several courses; (4) provide specific suggestions for teaching some topics.

Teachers will be well advised to consider these outlines and suggestions and their uses in relation to the kinds of students that populate their classes. Many classes will want to cover more material or the same material in more depth than is indicated. It is possible that, in a few circumstances, covering less may be advisable. The basic approach to all topics is that of contemporary mathematics. Even so, it is anticipated that within a very few years progress in the adoption of modern mathematics programs will require a review and revision of this state-level guide to bring it in line with contemporary content and practices.

Statement of Belief

The perspective to be gained from the gross attributes of society at present and the changes occurring in them can help us comprehend the need for certain emphasis in education in general and the study of mathematics in particular.

One of the distinct and important factors contributing to the great explosion of knowledge which has been taking place is the over-all revolutionary advance in the uses of mathematics. Theoretical mathematicians have never before in history produced more new ideas, new theories, and new potential for breakthroughs in all branches of science. The developments in the physical sciences are creating demands for new interpretations and uses of mathematics. Of possibly even greater significance in this revolution are the demands which are coming from the new users of mathematics.

The life sciences, the schools of business administration, and the social sciences are increasing their demands for the use of mathematical principles and techniques. The biologists are applying information theory to studies of inheritance; men from business and industry use the techniques of operations research in scheduling production and distribution; psychologists and sociologists make many uses of both elementary and the more sophisticated properties of modern statistics; the analysts of human behavior are finding major assistance in the principles and properties of game theory.

This revolution in the interpretation of the role of mathematics as an important element of our social structure is closely related to the evolution of the electronic computing machine, but it is even more firmly woven into the fabric of our social order.

The Need for Curriculum Change

The many new uses of mathematics not only have increased the demand for mathematicians but also have placed new emphasis on the type of training needed. Even the programs for "minimum essentials" in mathematics at different levels of attainment need to be subjected to critical reconsideration. This is true whether the evaluation is made in the context of the lay user of mathematics or the individual looking toward professional uses of mathematics. Thus we find that following a half century of relative stability the content of the mathematics program at all levels of instruction needs to be changed.

> Today the fastest-growing and most radically changing of all the sciences is mathematics. It is the only branch of learning in which all the major theories of two thousand years ago are still valid, yet never before has there been such a flood of fresh ideas. New branches of mathematics, like game theory, are beginning to yield remarkable insights into human relationships that scientists have never before analyzed precisely. Old branches, like probability theory, are being applied to such fresh areas as traffic flow and communication. And space flight challenges mathematicians to invent new navigational techniques far more complex than those that now guide ships and airplanes.[1]

The unprecedented demand for mathematicians, their need for a new and more intensive training program, and the basic mathematical requirements of our technological age combine to pose a very difficult curriculum problem that demands the thoughtful attention of those whose concern it is to provide the most effective program in mathematics for our secondary schools. This curriculum must guarantee an appropriate minimum program for every educable individual—whether slow learner, average pupil, or mathematically gifted.

Because of this we must focus our attention on change; for curriculum development implies change. We must show courage, for change may represent a threat to long-established values. Therefore, to reform our curriculum there is need first to reform ourselves.

The Need for Change in Point of View

The new uses of mathematics place great emphasis on its basic structure and on its function as a language in terms of which theories and hypotheses can be precisely formulated and tested. Rather than the manipulation of formulas and equations, the measurements of configurations, and the performance of computations, the principal contribution of mathematics is fast becoming the construction of mathematical models of events, whether actual or hypothetical.

Mathematical systems are man made. They evolve as models for the representation and interpretation of the physical universe. Thus, the physical universe provides a basis for pupil discovery and understanding of mathematical systems. At all levels of instruction more emphasis should be placed upon pupil discovery and reasoning, reinforced by greater precision of expression. When there is a basis of concrete experience, abstraction not only adds insight to the experience but also shows gradually where the topics of mathematics fit into a unified whole.

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[1] Boehm, George A. W. *The New World of Math.* New York: The Dial Press, 1959.

The Role of the Teacher

The main role of a teacher is to give direction to the learning that is to take place under his guidance, to encourage and motivate his students to learn, to plan and supervise situations by means of which learning becomes a thrilling experience for the students, and to help students evaluate their accomplishments. We believe the teacher is doing his best teaching when he plans for pupil discovery of ideas and encourages his students to do original or creative thinking. Also teachers have the responsibility of not only introducing new content but also of reinforcing and strengthening concepts and skills previously taught, and of laying the groundwork for the development of concepts at a later date.

Goals of Instruction

The goals of instruction may be listed as follows:

1. Insight into the nature or structure of mathematics
2. Mastery of skills in mathematical processes
3. Insight into the role of mathematics in the affairs of today's citizen as well as the affairs of the scientist and the engineer
4. Development of favorable attitudes and habits of work and thought through:
    a. Systematic and effective habits of study
    b. Independent work habits
    c. Appreciation of thoroughness and accuracy
    d. Habits of logical thought
    e. Curiosity for exploring and enthusiasm for mathematics.

Selection of Content

Based on the goals of instruction, content should be in terms of:

1. The extent to which it is used in understanding the nature of mathematics and in becoming prepared for further study of mathematics
2. The extent to which it is needed in analyzing and interpreting one's environment
3. The level of mathematical competence, native ability, and interests of the students.

CHAPTER 1

Fundamentals of Mathematics

THE VARIOUS LEARNINGS in a mathematics classroom may be grouped into the following four categories:

1. *Insights into the structure or concepts of mathematics.* Mathematics has a framework or structure; it is not a mass of isolated facts. Around a few basic principles most of the ideas of mathematics, which might at first seem unrelated, can be grouped. When mathematics "makes sense" to students, these students usually display greater interest in the subject and show greater retention of both concepts and skills learned.
2. *Skills in performing certain manipulative processes.* Care should be taken to provide adequate and appropriate practice exercises to help students gain proficiency and confidence in the processes of mathematics. In general, such practice exercises should follow experiences which students have had in investigating the significance and understandings of the processes. When the arithmetic processes are taught so as to emphasize their relatedness, learning becomes more effective. We have, in the past, been in such a hurry to push students into the manipulative phases of mathematics that many students have become "symbol pushers" with little or no insight into what the symbols mean. For example, when we have taught the measurement of area, have we not within the first hour—yes, within the first first ten minutes—of instruction led our students into the rule of "multiplying the length by the width to find the area of a rectangle" before these students understood the concept of area?
3. *Competence in using mathematical concepts and processes in solving problems.* We must breathe life into abstract mathematical symbolism. The teacher has the very difficult task of setting up problem situations that are interesting,

significant, and challenging to his students. Then he must allow the students to estimate, conjecture, try methods, verify results, and learn from his errors as well as from his successes. What types of problems are most interesting to junior high school students? The most interesting types may come from astronomy, social studies, and earth science rather than from business mathematics which has been the main source of problems in the past. Certainly, a variety of situations should be used.

4. *Desirable habits of work and thought.* The by-products of mathematics instruction may be fully as valuable for the seventh and eighth grade students as the skills taught or the concepts learned. Classrooms in which students' creative talents have been awakened, interests have been kindled, and curiosity has been aroused, are usually those classrooms in which there is an atmosphere of trying to find answers to questions—in which students are learning how to learn. These learnings are difficult to measure. Experienced teachers may also not be in complete agreement about how to promote desirable attitudes toward and habits of learning.

A few brief years ago the emphasis in mathematics instruction was on the second objective—teaching a mass of unrelated facts and processes. The emphasis is shifting today. More and more teachers are emphasizing the first objective—having their students discover mathematics as a meaningful structure. As the pendulum swings from the second to the first objective, a word of caution seems in order lest teachers lose sight of all four categories of objectives. Research data (particularly since 1945) seem to support a balance of emphasis between the four categories.

Teaching Suggestions

Students at the junior high school level are generally very active, outspoken, and differing in interests, many of which are transitory. Teachers who have been effective with these students offer the following helpful teaching suggestions:

1. Develop a sense of humor and a real enthusiasm for mathematics and for young people.
2. Raise questions. Encourage students to wonder why and to ask questions. Make the classroom a place for finding an-

swers. Teachers should not be too eager to supply answers; questions left unanswered for a few days often create school-wide interest.

3. Have a variety of reference materials and visual materials which may be used by students to explore ideas on their own. Gear the content and activities so as to challenge students of all levels of competence and of different interests.
4. Stress the significance of what is being studied.
5. Help students to give specific or concrete examples to the general and abstract symbolism of mathematics. Help them to visualize and to estimate answers to questions.
6. Stress the conceptual aspect of mathematics.
7. Make a habit of analyzing the thinking of youngsters. Study the art of asking questions.

Selection and Organization of Content

The following outline of topics comprises the mathematical concepts and skills considered basic for all students.

The selection of topics, the level of abstraction, and the speed with which these topics are treated will depend upon the amount of emphasis given to the structure of arithmetic in the elementary school program, the competence of the students, and the effectiveness of the teacher. It is anticipated that as the elementary school program becomes more mathematically oriented that many of the topics listed below will have to be treated in a review fashion, taught at a higher level of abstraction, or even eliminated from the junior high school curriculum. (The topic of whole numbers in the outline is an example.)

If a class is composed of the upper five to eight per cent (in mathematics ability on national norms), it may be able to complete many of the topics in less than two years, eliminate some aspects of some of the topics which are already familiar to them, and treat each of the topics at a more advanced level of abstraction.

It is anticipated that the lowest 40 per cent of the junior high school students (judged on the basis of mathematical competence) should spend a minimum of three years on the topics in the outline. Generally speaking, these students must proceed at a slower pace, use more concrete illustrations, and operate on a lower level of abstraction than the more advanced students. It is assumed that applications would come from geometry and science and not be limited to consumer or pocketbook examples. It is recommended, however, that skills and applications not receive such great emphasis that the other categories of learning are neglected for these students. Mathematics must also "make sense" to them. They, too, can receive the thrill of doing creative thinking.

The ninth grade general mathematics course represents the content of the third year of study of "Fundamentals of Mathematics," and its topics may be derived from this outline. Opportunity, of course, must be allowed in the general mathematics course for basic concepts to be reviewed and strengthened and for student errors in computational skill to be diagnosed and remedied.

In some schools a course in consumer mathematics is offered in the eleventh or twelfth grade for the non-college bound students who need two units in mathematics. There is much dissatisfaction with such courses which generally stress the same drill exercises and applications for all students. Some teachers are recommending that such a terminal course be built around mathematical principles with the students being given selective drill following a diagnosis of their competence. Other teachers are experimenting with more individualization of instruction with emphasis on individual or small group projects. This committee believes that the essential mathematical content of a consumer mathematics course could be found in the outline of "Fundamentals of Mathematics".

It seems, furthermore, that topics may be treated in a classroom in any one of three stages:

1. *Background stage*—an intuitive approach which lays the background for a more thorough and a more abstract treatment to come later.
2. *Developmental stage*—the drawing of abstractions from students' intuitive experiences, the relating of these abstractions to other abstractions which students have previously made, and the development of proficiency in using these abstractions in solving problems.

3. *Maintenance or reinforcement stage*—the review of concepts or procedures previously developed, which may involve a look from a different viewpoint by the student who has studied topics in the meantime which are related to the idea being reviewed.

Because topics might be treated at any of these three levels, many of the ideas in our outline can be studied at more than one grade level. Again, the emphasis and level of treatment depend upon the background of the students, and this background varies a great deal. *In view of these facts, the committee decided, instead of listing the topics by grades in the junior high school, to star (*) those topics which were more advanced or more complicated.*

Certain topics have recently been introduced into the junior high school curriculum because these topics can be used to help students gain a better insight into the structure of mathematics. Non-decimal numeration systems and set language are examples. The purpose of studying non-decimal systems of numeration should not be the development of computational skills in those systems, but instead it should be the further development of the student's insight into the structure of our own decimal system of numeration.

The concept and language of sets can be of great value in defining and giving meaning to such ideas as the following:

1. counting
2. categories of numbers—whole, non-negative rationals, integers, negative rationals, and reals
3. operations with numbers
4. variable—its replacement set and its solution set
5. greatest common factor and least common multiple
6. geometric figures such as line, line segment, angle.

The topic of sets should not be studied in great detail and then left without use during the rest of the year. It should be introduced and expanded upon as needed to contribute to the understanding of the aforementioned ideas.

The following outline contains a variety of approaches. In some instances, merely a listing of topics was considered most appropriate; in other instances, a great deal of space was taken to explain or to illustrate an idea.

Outline of Topics

I. Concept of Whole Numbers and Operations with Them

A. The Collection Idea

1. A collection of things with a well-defined feature or property is referred to as a *set*. The things that belong to a set are called *members* of the set. A finite set is one whose members can be counted or listed. Sets which are not finite are said to be infinite.
2. A whole number may be associated with a finite set. When the members of two sets can be paired, *i.e.*, placed in a one-to-one correspondence, we say both sets have the same *cardinal* number. The symbol that we write as a name for a number is called a *numeral*.
3. By partial listing, the set of natural numbers is:

   $\{1, 2, 3, \ldots\}$
4. The set of whole numbers: $\{0, 1, 2, 3, 4, \ldots\}$
5. Interpretation of whole numbers in terms of cardinality of sets.

   $$\begin{array}{lll}
   n\{\} & & =0\\
   n\{\triangle\} & =n\{1\} & =1\\
   n\{\triangle, \mathrm{O}\} & =n\{1,2\} & =2\\
   n\{\triangle, \mathrm{O}, \square\} & =n\{1,2,3\} & =3\\
   n\{\triangle, \mathrm{O}, \square, \times\} & =n\{1,2,3,4\} & =4
   \end{array}$$

   etc.

   The symbol "$n\{\triangle, \mathrm{O}, \square\}$" is read "the cardinal number of the set $\{\triangle, \mathrm{O}, \square\}$".
6. The set of natural numbers is *ordered*. When these numbers are used to indicate the order of things, the numbers are used in the *ordinal* sense.

B. Interpretation of operations with whole numbers in terms of set operations.

1. *Addition:* If $n(A) = a$ and $n(B) = b$, then $n(A \cup B) = a + b$, providing A and B are disjoint sets.

2. ***Subtraction:*** **$a - b = \square$ is equivalent to $\square + b = a$. Subtraction is the inverse of addition; it is the process of finding one of the two addends, when the other addend and the sum are known.**

3. ***Multiplication:*** **$a \cdot b$ represents the number of elements in a disjoint sets of b elements each, for example, the number of elements in a rectangular array of *a* rows and *b* columns. In $a \cdot b$ the a and b are called factors; $a \cdot b$ also represents the sum of *a* identical addends of *b*.**

4. ***Division:*** **$a \div b = \square$ where $b \neq 0$, is equivalent to $\square \cdot b = a$ or $b \cdot \square = a$. Division is the inverse of multiplication; it is the process of finding one of the two factors when the other factor and the product are known. Whereas multiplication is associated with the process of joining a given number of equivalent disjoint sets together and finding the number of elements in the resulting set, division is the process of separating (or partitioning) a set of elements into equivalent disjoint subsets. $12 \div 3 = \square$ being equivalent to $\square \cdot 3 = 12$ asks for the number of disjoint subsets of 3 elements each in a set of 12 elements. $12 \div 3 = \square$ being equivalent to $3 \cdot \square = 12$ asks for the number in each subset when a set of 12 elements has been separated (partitioned) into 3 equivalent disjoint subsets.**

C. Properties of Whole Numbers

1. ***Closure:*** **If a and b are whole numbers, then $a + b$ is a unique whole number; $a \cdot b$ is also a unique whole number. The set of whole numbers is not closed under either subtraction or division, however.**

2. ***Commutativity:*** **If a and b are whole numbers, then $a + b = b + a$; $a \cdot b$ is also equal to $b \cdot a$. Neither subtraction nor division of whole numbers is commutative.**

3. ***Associativity:*** **If a, b and c are whole numbers, then $(a + b) + c = a + (b + c)$; also $(a \cdot b) \cdot c = a \cdot (b \cdot c)$. Neither subtraction nor division of whole numbers is associative. Thus $8 + 7 + 3$, without grouping symbols, has only one meaning; while $8 - 7 - 3$ is ambiguous without grouping symbols.**

4. *Distributivity:*

   a. Of multiplication over addition and subtraction. If a, b, and c are whole numbers, then $a \cdot (b + c) = (a \cdot b) + (a \cdot c)$ and $(b + c) \cdot a = (b \cdot a) + (c \cdot a)$. $a \cdot (b - c) = (a \cdot b) - (a \cdot c)$.

   b. Of division over addition and subtraction (right hand only):
   $(b + c) \div a = (b \div a) + (c \div a)$
   $a \div (b + c) \neq (a \div b) + (a \div c)$.
   $(a - b) \div c = (a \div c) - (b \div c)$
   $c \div (a \cdot b) \neq (c \div a) - (c \div b)$.

5. Special properties of 1. For any whole number n,

   a. $n \cdot 1 = 1 \cdot n = n$

   b. $\frac{n}{1} = n$

   c. $\frac{n}{n} = 1$, providing $n \neq 0$.

6. Special properties of 0. For any whole number n,

   a. $n + 0 = n$

   b. $n - 0 = n$

   c. $\frac{0}{n} = 0$, providing $n \neq 0$

   d. $\frac{n}{0}$ is not defined.

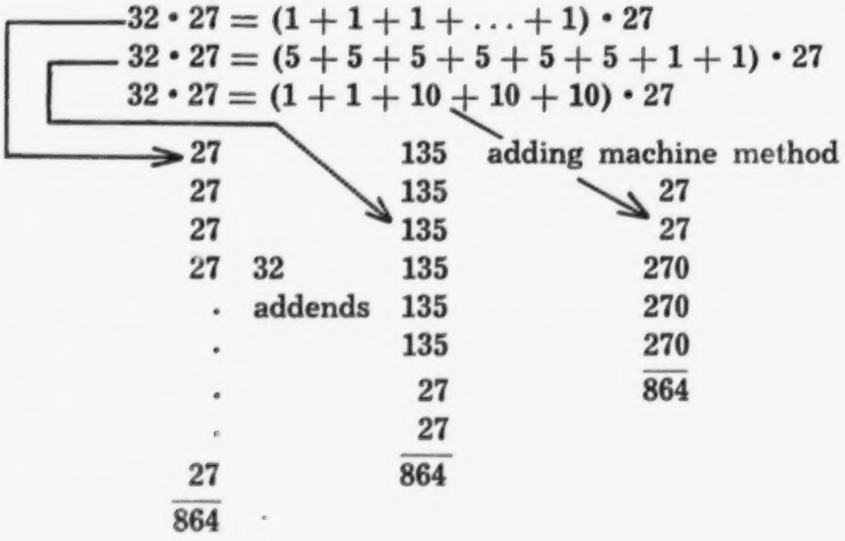
D. Development of algorisms for operations with numbers in multidigit numerals.

1. The product of a number by 10^n is equivalent to that number with digits moved n places to the left. For example,

$$\begin{aligned} 100(364) &= 10^2[3(10)^2 + 6(10) + 4(1)] \\ &= 3(10)^4 + 6(10)^3 + 4(10)^2 \\ &= 36400 \end{aligned}$$

2. Distributive principle in multiplication. Some examples follow:

$$\begin{aligned} 3 \cdot (21) &= 3(20 + 1) \\ &= 3 \cdot 20 + 3 \cdot 1 \\ &= 60 + 3 \\ &= 63 \end{aligned}$$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32 • 27 = (5 + 5 + 5 + 5 + 5 + 5 + 1 + 1) • 27
32 • 27 = (1 + 1 + 10 + 10 + 10) • 27
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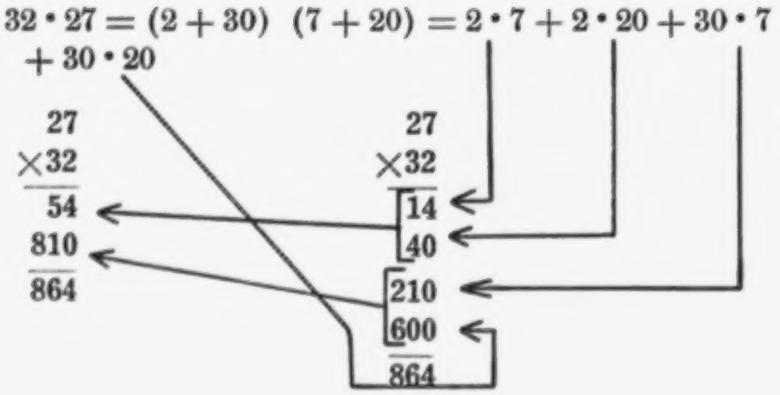
32 • 27 = (2 + 30) (7 + 20) = 2 • 7 + 2 • 20 + 30 • 7
+ 30 • 20
27
×32
54
810
864
27
×32
14
40
210
600
864

3. Distributive principle in division. Separate the dividend into addends which are multiples of the divisor.

$$\frac{864}{27} = \frac{270 + 270 + 270 + 27 + 27}{27} = 10 + 10 + 10 + 1 + 1 = 32$$

$$\frac{864}{27} = \frac{540 + 270 + 54}{27} = 20 + 10 + 2 = 32$$

$$\begin{array}{rl} \left.\begin{array}{r} 2 \\ 10 \\ \underline{20} \end{array}\right] & = 32 \\ 27\,\overline{)\,864} & \\ \underline{-540} & \\ 324 & \\ \underline{-270} & \\ 54 & \\ \underline{-54} & \end{array} \qquad\qquad \begin{array}{r|l} 27\,\overline{)\,864} & \\ -540 & 20 \\ \hline 324 & \\ -270 & 10 \\ \hline 54 & \\ -54 & 2 \\ \hline & 32 \end{array}$$

E. Estimating results:

1. Numbers can be rounded to 1 or 2 significant digits and these rounded numbers used in estimating. The symbol “$\approx$” means “approximately equal to.”
2. In multiplication, multiplying one factor by 10^n and dividing a second factor by 10^n does not change the product.

$$\begin{aligned} 314.2 \times .089 &\approx 300 \times .09 \\ &\approx 3.00 \times 9 \\ &\approx 27 \end{aligned}$$

3. In division, multiplying (or dividing) both numerator and denominator by the same non-zero number does not change the quotient.

$$\frac{3{,}826{,}519}{790{,}206} \approx \frac{4{,}000{,}000}{800{,}000} = \frac{40}{8} = 5$$

$$\frac{.03016}{.00612} \approx \frac{.03}{.006} = \frac{30.}{6.} = 5$$

It is convenient to make the divisor a whole number between 1 and 9.

$$\frac{.0042}{.806} \approx \frac{.040}{8} = .005$$

F. Divisibility, factors, multiples, and primes

1. If a, b, and c are whole numbers, a is said to divide b, if and only if $a \neq 0$, and there exists a whole number c such that $a \cdot c = b$; *a* and *c* are factors of b; in other words, divisors are non-zero factors.

2. All numbers have a set of unique divisors:

Number	Set of divisors
1	{1}
2	{1,2}
3	{1,3}
4	{1,2,4}
etc.	
0	{1, 2, 3, 4, 5, 6, 7, . . .}

   All numbers divide 0. Only one number divides 1. Numbers which have a set of two divisors are called *primes*; numbers having 3 or more divisors are called *composite* numbers; numbers greater than 1 having an odd number of divisors are *square* numbers.

3. The set of multiples of natural number *a* is the following:

   $\{1 \cdot a, 2 \cdot a, 3 \cdot a, 4 \cdot a, \ldots\}$

4. Prime numbers and their significance
   a. Sieve of Eratosthenes
   b. Unique Factorization Property of natural numbers
      {1, 2, 3, 4, 5, . . .}
      Every composite natural number can be written as the product of one and only one set of primes.
   c. Greatest common factor (GCF) of 2 or more natural numbers
   d. Least common multiple (LCM) of 2 or more natural numbers and its relation to *common denominator* of fractions.
   e. Two (or more) natural numbers are *relatively prime* providing their greatest common factor is 1.

5. Properties of odd and even numbers

6. The Euclidean Algorithm
7. Digital tests for divisibility by 2, 3, 4, 5, 6, 8, 9

*8. Discovery of number theorems:
   - a. A number written in base ten numerals less the sum of its digits is divisible by 9.
   - b. The sum of the first n odd whole or natural numbers is equal to n^2.
   - c. The product of the GCF and LCM of two numbers is equal to the product of the two numbers.
   - d. The prime numbers greater than 3 are either 1 less than or 1 greater than a multiple of 6.
   - e. The set of odd numbers is closed under multiplication.

II. Numeration Systems

A. Grouping of objects "to be counted" is done to economize on the use of symbols.
   1. The base of grouping is arbitrary.
   2. The principle of grouping can be extended to larger and larger size groups.
   3. One can use "counters" on an abacus to represent how many bundles of each size we have.
   4. Computing may be done with these counters.

B. Basic types of structure of systems of numerals which might be used to represent both the sizes of these bundles and the number of bundles of each size.
   1. Repetitive schemes use different symbols for each different bundle size and *repeat* each symbol to indicate how many bundles of each size. Probably the simplest system of this type of historical importance is that of the Egyptians. One might learn how to use these numerals for enumeration and even for computation.
   2. Positional schemes use the *position* of a symbol to show bundle size and the choice of symbol to show the number of bundles of each size.
      - a. Hindu-Arabic is the prime example.

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* Advanced topic

b. This system uses a zero symbol.

c. The additive idea is implied in the system as is shown below in the expanded or polynomial forms of the numeral.

(1) $4273 = (4 \times 1000) + (2 \times 100) + (7 \times 10) + (3 \times 1)$

(2) $4273 = 4 \times 10^3 + 2 \times 10^2 + 7 \times 10^1 + 3 \times 10^0$

C. (Optional) Other numeration systems of historical significance may be studied, such as Roman, Greek, Babylonian, Mayan, Hebrew, and Chinese. Students may be interested in creating an original system of numerals.

D. Non-decimal Numeration

1. Bases less than ten, *e.g.*, binary and quinary
2. Bases greater than ten, *e.g.*, twelve
3. Changing from base ten to other bases and from other bases to base ten, *e.g.*,

   302_{five} means $(3 \times 5^2) + (0 \times 5^1) + (2 \times 5^0)$

*4. Computation with non-decimal numerals

5. Relating non-decimal bases to units of measure, *e.g.*,

   12 days $= 15_7$ days $= 1$ week $+ 5$ days
   53 inches $= 45_{12}$ inches $= 4$ feet $+ 5$ inches

E. Special ways of writing numbers using exponents—Scientific Notation—A number is in scientific notation (base ten) when it is written as

$a \times 10^b$ where $1 \leq a < 10$ and b is an integer.

Examples: weight of earth $\approx 1.3 \times 10^{25}$ lbs
population of U. S. $\approx 1.8 \times 10^8$ people
number of drops of water in Atlantic Ocean $\approx 8 \times 10^{24}$ drops
radius of hydrogen atom $\approx 1.74 \times 10^{-10}$ feet

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* Advanced topic

III. Concept of Non-Negative Rational Numbers (Numbers of Arithmetic) and Operations with Them

A. Interpretation

In the historic development of our numeral symbols, the need for representing a part of a unit by using the previously known counting numerals was met by using the fractional numeral forms. Thus, "$\frac{3}{4}$" was interpreted as "three of four equal parts."

Since this initial development and use, many other interpretations have arisen. One author describes four principal meanings of interpretations of these number pair symbols.

1. An element of a mathematical system[1]
2. The division of two naturals or integers
3. The fraction or partition idea
4. The ratio or rate pair idea

The four interpretations

Number System	*Division*
$n \cdot x = m$ $x = \frac{m}{n}$	$\frac{m}{n} = k = m \div n$
The multiplicative inverse	Decimal representation
Partition (Fractions)	*Rate Pair (Ratio)*
Measurement Magnitude m of n equal parts.	Relative comparison of quantities or sets. "m to n" per cent.

B. Association of the Rationals with the Number Line

We can define a one-to-one correspondence between the rational numbers and a sub-set of the points on a line by associating to each rational number, the point on the *number line* whose *distance* to the right of some fixed point (the zero point) is that rational number.

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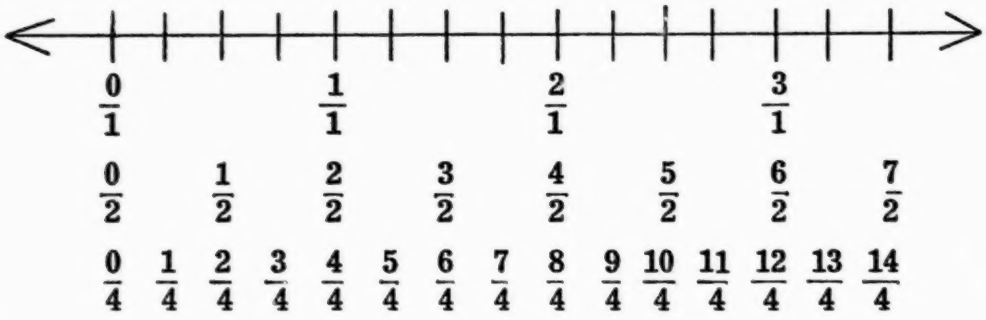
[1] Peterson, John and Hashisaki, Joseph. *Theory of Arithmetic.* New York: John Wiley, 1963, p. 151 f.

One such representation which also shows that there are many names for a rational number is as follows:


$\frac{0}{1}$ $\frac{1}{1}$ $\frac{2}{1}$ $\frac{3}{1}$
$\frac{0}{2}$ $\frac{1}{2}$ $\frac{2}{2}$ $\frac{3}{2}$ $\frac{4}{2}$ $\frac{5}{2}$ $\frac{6}{2}$ $\frac{7}{2}$
$\frac{0}{4}$ $\frac{1}{4}$ $\frac{2}{4}$ $\frac{3}{4}$ $\frac{4}{4}$ $\frac{5}{4}$ $\frac{6}{4}$ $\frac{7}{4}$ $\frac{8}{4}$ $\frac{9}{4}$ $\frac{10}{4}$ $\frac{11}{4}$ $\frac{12}{4}$ $\frac{13}{4}$ $\frac{14}{4}$

The above representation reveals that the set of whole numbers constitute a subset of the non-negative rationals. In problems such as the following use is made of this fact.

1. $2+\frac{1}{2}=\frac{4}{2}+\frac{1}{2}$ 2. $3\times\frac{5}{8}=\frac{3}{1}\times\frac{5}{8}$

C. Meaning of Rational Numbers in Terms of Concrete Situations for Each Interpretation

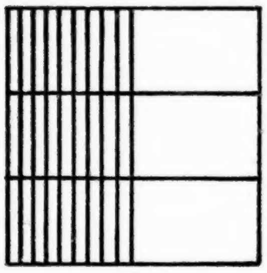
An abundance of concrete situations should be used with the study of the rational numbers. These concrete situations should relate the rational numbers to everyday settings, such as science, sports records and events, various school experiences, social sciences, and carpenter-type building situations as well as to the more abstract properties such as denseness, etc.

D. Principle of Equivalent Rational Numbers

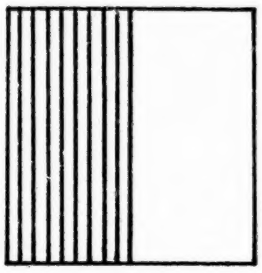
1. Formally: Two rational numbers

$\frac{a}{b}$ and $\frac{r}{s}$ are said to be equal if and only if $a \cdot s = b \cdot r$

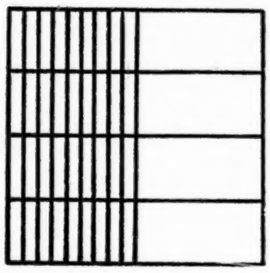
2. Using rectangular regions as models, equivalent rationals can be exhibited as shown below.



$\frac{3}{6}$



$\frac{1}{2}$



$\frac{4}{8}$

3. Simplest fractional numeral: The simplest fractional numeral for a rational number is that numeral which has the smallest possible *whole* number for its numerator and the smallest possible *natural* number for its denominator. The numerator and denominator will have no common factor except 1. Proceed as follows:

$$\frac{8}{12}=\frac{2\times 4}{3\times 4}=\frac{2}{3}\times\frac{4}{4}=\frac{2}{3}\times 1=\frac{2}{3}$$

E. Behavior of Rational Numbers under the four operations

1. The binary operations of addition and multiplication are usually defined first, then subtraction and division develop from them.

They are defined as follows:

Where b and d $\neq 0$,

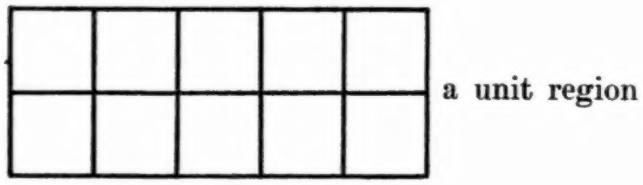
$$\frac{a}{b}+\frac{c}{d}=\frac{ad+bc}{bd}$$

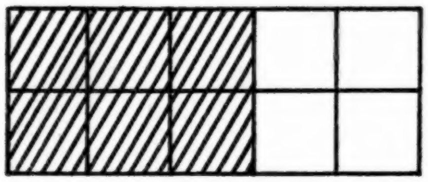
$$\frac{a}{b}\cdot\frac{c}{d}=\frac{ac}{bd}$$

2. Models used to interpret the operations. Example:

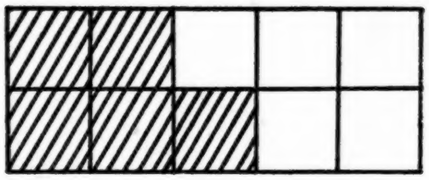
$$\frac{6}{10}+\frac{5}{10}=\frac{11}{10}$$

Regions

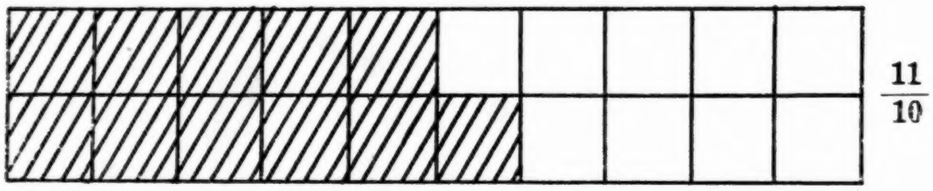

a unit region



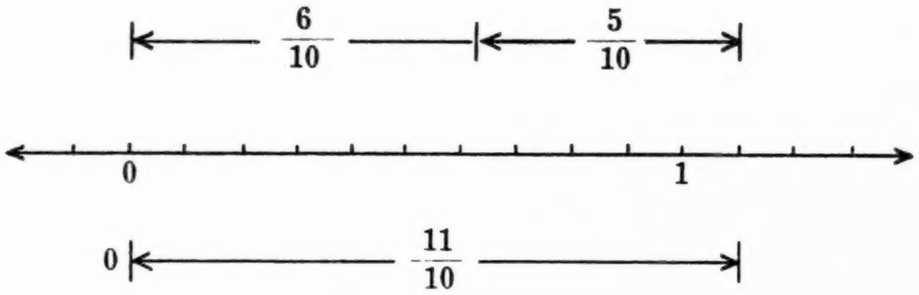
$\frac{6}{10}$



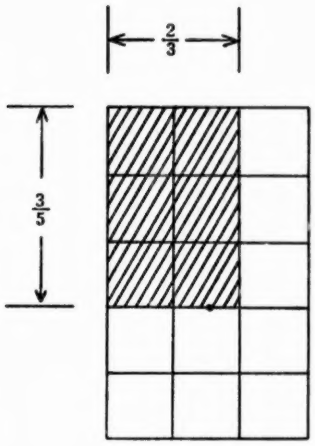
$\frac{5}{10}$



Number Lines:

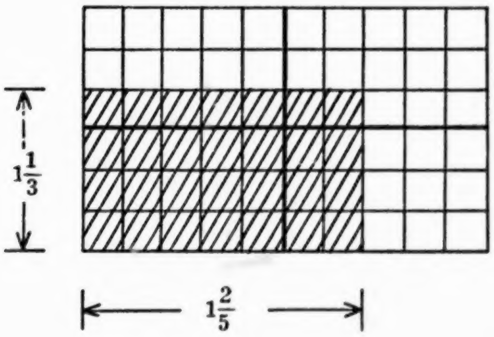

$\frac{6}{10}$
$\frac{5}{10}$
0
1
0
$\frac{11}{10}$

Example: $\frac{3}{5}\times\frac{2}{3}=\frac{6}{15}$


$\frac{2}{3}$
$\frac{3}{5}$

Example:

$$1\frac{1}{3}\times1\frac{2}{5}=\frac{4}{3}\times\frac{7}{5}=\frac{4\times7}{15}=\frac{28}{15}=1\ \frac{13}{15}$$


$1\frac{1}{3}$
$1\frac{2}{5}$

3. The two operations defined in (1) have these properties:
   a. Closure in the set of fractional numbers.
   b. Commutativity and associativity.
   c. Multiplication distributes over addition.
   d. Zero is the identity for addition and one is the identity for multiplication.
   e. Each non-zero fractional number has an inverse for multiplication, called the reciprocal. The product of a number and its reciprocal is one. The reciprocal can be named as $\frac{1}{\frac{a}{b}}$ or $\frac{b}{a}$.

   Division by a number can be interpreted as multiplication by the reciprocal, where b, r, and s $\neq 0$.

   $\frac{a}{b} \div \frac{r}{s}$ means $\frac{a}{b} \cdot \frac{s}{r}$

4. The common denominator for two fractional numbers is a *multiple* that is common to each of the denominators. The least common multiple is the least member of the set of common multiples.

   Example: $\frac{5}{6}$ and $\frac{1}{4}$

   Set of multiples of 6 = A = {6, *12*, 18, *24*, . . .}

   Set of multiples of 4 = B = {4, 8, *12*, 16, 20, *24*, . . .}

   A ∩ B = {12, 24, . . .}

   Thus, the least member of the intersection set is 12.

F. Relationship between the fractional forms

1. The mixed form: When we mean $66 + \frac{2}{3}$ we sometimes write $66\frac{2}{3}$ omitting the plus sign.

2. The place value or decimal numeral form
   a. Meaning: An extension of the base ten place value notation

$$\begin{aligned} 3+\frac{25}{100} &= 3+\frac{2}{10}+\frac{5}{100} \\ &= 3+2\left(\frac{1}{10}\right)+5\left(\frac{1}{100}\right) \\ &= 3.25 \end{aligned}$$

   b. Changing from fractional numeral form to decimal numeral form

(1) $\frac{3}{4}=\frac{3}{2\times 2}\cdot 1=\frac{3}{2\times 2}\cdot\frac{5\times 5}{5\times 5}=$

$\frac{3\times 5\times 5}{2\times 5\times 2\times 5}=$

$\frac{75}{100}=.75$ (exact representation)

(2) $\frac{5}{8}=5\div 8$, thus it is

$= .625$ (exact representation)

$$\begin{array}{r} .625 \\ 8\overline{)5.000} \\ \underline{4\,8} \\ 20 \\ \underline{16} \\ 40 \\ \underline{40} \end{array}$$

(3) $\frac{1}{3}=1\div 3=.33\frac{1}{3}$ (exact representation)

$=.333\ldots$ (exact representation)

$\frac{1}{3}\approx .333=\frac{333}{1000}$ (approximate representation)

$$\begin{array}{r} .333 \\ 3\overline{)1.000} \\ \underline{9} \\ 10 \\ \underline{9} \\ 10 \\ \underline{9} \\ 1 \text{ etc.} \end{array}$$

3. The algorisms for the decimal numeral forms
   a. Addition and subtraction
   b. Multiplication
   c. Division

***4. Repeating and terminating numerators**

a. Any fractional number (rational number) has a decimal representation which either terminates or is periodic and non-terminating.

When the denominator of a fractional number is not a power of 2 or 5, it will have a repeating numerator when expressed in decimal (base ten) notation.

$\frac{1}{4} = .25000\ldots$ (terminating)

$\frac{1}{6} = .1666\ldots$ (repeating)

b. Changing from repeating decimal numerals to a fractional numeral, consider

$$.2727\ldots = \frac{a}{b}$$

$$\begin{array}{rl} 10^2 \times .2727\ldots &= 27.2727\ldots \\ -\ 1\ \times .2727\ldots &= .2727\ldots \\ \hline 99\ \times .2727\ldots &= 27 \end{array}$$

$$\text{Thus}\quad .2727\ldots = 27 \times \frac{1}{99}$$

$$= \frac{27}{99} \text{ or } \frac{3}{11}$$

G. Using fractions in problem solving situations.

This material should be used throughout the development, relating fractional numbers to such situations as science, do-it-yourself home jobs, current events in sports, hobbies, school situations, and social sciences.

***H. Fractional numerals written with non-decimal bases**

1. $(2.13)_{\text{five}}$ means $2 + 1\left(\frac{1}{5}\right) + 3\left(\frac{1}{25}\right)$

2. $\left(\frac{1}{3}\right)_{\text{six}} = (0.2)_{\text{six}} = 2\left(\frac{1}{6}\right)$

3. $\left(\frac{1}{4}\right)_{\text{six}} = (0.13)_{\text{six}} = 1\left(\frac{1}{6}\right) + 3\left(\frac{1}{36}\right)$

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* Advanced topic

IV. Ratio

A. Concept of ratio

1. A simple ratio is an ordered pair of real numbers, the first number representing the count (or measure) of the first set (or thing) and the second number that of the second set. These two sets of things may be of the same type or of different types. For example, the ratio of the number of cars to the number of students may be 1023 : 14,467. The ratio of the number of centimeters to the number of inches in the measure of a line segment may be expressed approximately as 2.54 : 1.
2. Two ratios that express the comparison of the same two sets are called equivalent or equal ratios. $(a : b) = (c : d)$ if and ony if $a \cdot d = b \cdot c$.
3. Symbolism: $(a : b)$ and $\frac{a}{b}$ forms.

B. Uses of Ratios

1. Ratios are used to make a set-to-set comparison (rather than a 1-to-1 comparison) of two sets.
2. In many situations in which the comparison does not change, one may express two ratios as equivalent and in doing so will be able to find a missing component of one of the two equal ratios. Such an expression of the equality of two ratios is called a *proportion.*

C. Ratios with special second components

1. If the second component of a ratio is 100, the ratio is called a *per cent* $(5 : 100) = 5\%$. The three cases of ratio problems can be handled by the method of equal ratios. In all such problems we will write two equal ratios in which the second component of one of the ratios is 100. In other words, we will always get a proportion such as $(m : n) = (r : 100)$. The familiar case I, case II, and case III types of problems correspond to proportions in which the variable is in position m, r, and n, respectively. Example:
   a. What is 23% of 400?
      $(m : 400) = (23 : 100)$

b. 8 is what per cent of 12?
(8 : 12) = (r : 100)

c. 24 is 25% of what number?
(24 : n) = (25 : 100)

2. If the second component is 1, the ratio is called a *unit rate.* Examples:

   a. *Speed.* If I travel 55 miles in 1 hour, we speak of my rate of speed as 55 miles per hour or may write it as (55 : 1). If I travelled 150 miles in 2¾ hours, what was the average rate of travel?

      (150 : 2¾) = (x : 1)

   b. *Per unit cost.* We may compare the cost of two brands of a product by computing the per unit cost of each brand and comparing these two results.

   c. In many instances in which the second component is 1, the ratio is written as one number rather than a pair of numbers. Pi is the ratio of the measure of the circumference to the measure of the diameter of a circle and is approximately (3.14 : 1). We usually speak of pi, however, as 3.14.

D. Ratios are extremely useful. A few examples follow.

1. Conversion of units. The equivalent ratio method is quite valuable in converting a measure of one unit of measure to another one. Conversion ratios may be used. When dealing with centimeters and inches, the conversion ratio is (2.54 : 1); inches and feet (12 : 1); and miles per hour and feet per second (60 : 88).

   a. What is the length in inches of a meter stick?
      (100 : x) = (2.54 : 1)

   b. Hayes of FAMU ran the 100 yards in 9.1 seconds. What is the average speed in feet per second?
      (300 : 9.1) = (x : 1)
      x = 33

   c. What is Hayes' speed in miles per hour? (x : 33) = (60 : 88)

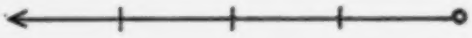
2. Similar figures. If two figures have the same shape, the ratio of the measures of two corresponding line segments is constant.
    a. One can use this principle to compute distances from a map.
    b. Use similar triangles to compute distances.
    c. Compute the size of animals from pictures in a dictionary for which a ratio is also provided.
3. Ratios are used in science—such as specific gravity and the formulas for chemical compounds.
4. Ratios are used in music.
5. Ratios are used in economics—such as price index and rate of growth.
6. Mixture problems also make use of ratios which may involve more than two components.

V. Integers and Rationals

A. Definition and Interpretation

1. The set of integers consists of the set of natural numbers, zero, and, for each natural number n, the *opposite* or *negative* of n, written as "^-n." Also $n + (^-n) = 0$, thus ^-n may be called the *additive inverse* of n. In set notation the set of integers is $\}. . . , {}^-n, ... {}^-3, {}^-2, {}^-1, 0, 1, 2, 3, . . . n, . . .\}$.
2. An integer is not a counting number which is an answer to the question "How many?" An integer can be interpreted as:
    a. The *status* of some measure, indicating where the measure is with respect to some chosen reference or zero position; *e.g.*, temperature of $^-15°$; elevation of $^+1420$ feet; assets of $^-\$10,000$: time, as $^-10$ seconds from blast off; the point corresponding to $^+5$ on the number line.
    b. A vector or change in which both the distance (amount) and direction must be indicated, e.g., $^+5$ as a trip 5 miles to the east; $^-3$ as a 3 lb. loss of weight;

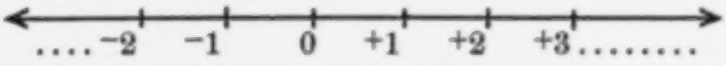
$^{+}10^\circ$ increase in temperature; $^{-}5$ yd. as a 5 yard penalty in football or $^{-}4$ as corresponding to the vector.



c. The difference between two natural numbers; *e.g.*, $^{-}2$ can be thought of as corresponding to equivalence class

$\{(1-3), (2-4), (3-5) \ldots\}$.

3. The integers can be placed in a one-to-one correspondence with equally-spaced points on a line:



4. The set of rational numbers is the union of the set of numbers of arithmetic (fractional numbers) and the set of additive inverses for these fractional numbers. Thus defined, the set of integers is a sub-set of the set of rationals.

5. The set of integers can be partitioned into three mutually disjoint subsets: positive integers, zero, and negative integers. The rationals can be partitioned similarly.

B. Properties of Integers (and Rationals)

1. Trichotomy Relation: If a and b are two integers (or two rationals), exactly one of the following relations exists:

   a. a = b

   b. a is greater than b

   c. a is less than b

2. Denseness of the set of rationals: Given any two rationals, a and b, say $a < b$. There always exists a rational number c such that $a < c < b$. This property does not hold for the set of integers.

3. Absolute value of integers (and rationals)

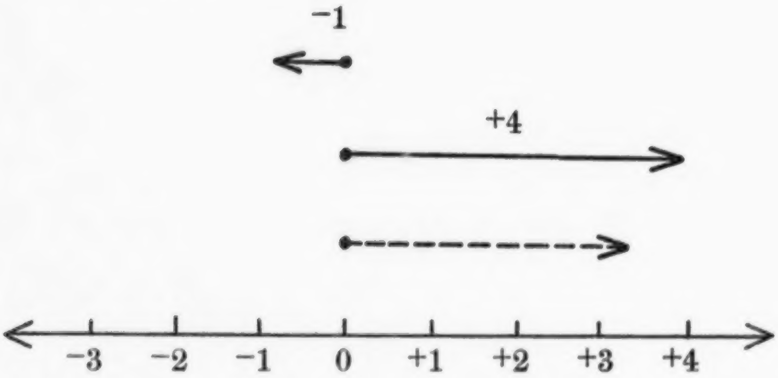
   a. The absolute value of an integer (or of a rational) n is represented by the symbol "$|n|$."

b. $|n| = n$, when $n \geq 0$
$|n| = -n$, when $n < 0$

c. Interpretation as the distance of a point from the zero point on the number line.

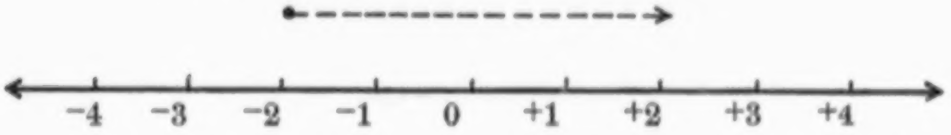
*C. Addition and Subtraction

1. The definition of addition and subtraction for integers and for rational numbers shall be defined so that the properties found for the set of natural numbers and the set of numbers of arithmetic (the fractional numbers) shall continue to hold.



The sum ${}^{-}1 + {}^{+}4$ can be represented on the number line as a vector of ${}^{-}1$ followed by a vector ${}^{+}4$. The sum of these two vectors is the vector ${}^{+}3$.[2]

The subtraction of arithmetic numbers $8 - 5$ can be thought of in terms of addition: "What number must be added to 5 to get 8?" In like manner, the subtraction of two integers $A - B$ can be thought of as "What vector will take us from B to A on the number line?"



${}^{+}3 - {}^{-}2$ can be thought of as "What number must be added to ${}^{-}2$ to get ${}^{+}3$?" or "What trip must one take along the number line in order to move from ${}^{-}2$ to ${}^{+}3$?"

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* Advanced topic

[2] Attention is called to the treatment of integers in *Experiences in Mathematical Discoveries* of NCTM.

2. Formally:

   For any numbers of arithmetic m and n

   a. $^{+}m + {}^{+}n = {}^{+}(m + n)$

   b. $^{-}m + {}^{-}n = {}^{-}(m + n)$

   c. $^{-}m + {}^{+}n = -(m - n)$, when $m > n$.
   $^{-}m + {}^{+}n = n - m$, when $m < n$.

3. Subtraction is the opposite of addition, thus it is defined as adding an additive inverse. For any rational numbers (or integers) m and n: $m - n = m + ({}^{-}n)$.

4. Properties for addition of rational numbers (and integers)

   a. Closure, commutativity, associativity

   b. Zero is the identity and each number has an additive inverse.

*D. Multiplication and Division

1. As was true for addition and subtraction so for multiplication and division, the operations are defined so that the properties discussed above will continue to hold. The only additional property is that of distributivity. It also holds for rational numbers (and integers).

2. Formally:

   For any numbers of arithmetic m and n

   a. $^{-}m \cdot {}^{-}n = m \cdot n$

   b. $^{-}m \cdot n = -(m \cdot n)$

3. Division is defined as multiplication by the reciprocal. If

   $\frac{r}{s}$ and $\frac{m}{n}$ represent rational numbers, then:

   $\frac{r}{s} \div \frac{m}{n} = \frac{r}{s} \cdot \frac{n}{m}$, where $\frac{m}{n}$ is not zero.

4. Properties for multiplication of rational numbers (and integers)

   a. Closure, commutativity, associativity.

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* Advanced topic

b. One is the identity number and each number, except zero, has a multiplicative inverse, called its reciprocal.

c. Multiplication distributes over addition.

E. Using positive and negative rationals in specific problem solving situations. One such example is that of the lever; others may relate to elevations, temperatures, time, and profit-and-loss.

VI. Mathematical Statements

A. The language of mathematics consists of symbols (alphabet), ways of joining symbols together (words and phrases), operation with numbers (grammar or syntax), and statements of relations between numbers (sentences).

B. Symbols

1. Symbols are invented to make communication easier in mathematics.
2. Examples of symbols: $2 + 3$, $=$, $<$, $\{\ \}$, $+$, $|\,n\,|$.
3. Distinction between the symbols and the ideas (made but not emphasized) such as: a numeral is any symbol that represents a number; we write numerals, we think about numbers: for the absolute value of any real number n the symbol "$|\,n\,|$" is used.
4. History of mathematical symbols with emphasis on numerals.

C. Number phrases and sentences

1. Exampes of number phrases:

   $8n - 6$, $3 \times {}^{+}5$, $5x^2 + 3y$, $3 \cdot 4 + 2$

   Each number phrase is the name for a number, *e.g.*, $3 + 8$ names 11 and $3x - 2$ names a different number for each replacement for x.
2. Number sentences state a relationship that exists between two numbers
   such as $3 \cdot 4 + 2 = 14$ and $3x + 10 > 5$.
3. Number sentences (both equality and inequality) are of the two types

a. Open—equivalent to an interrogative or incomplete sentence using a symbol, called a variable. Examples:

$x > 5$ What number is greater than five?

$n + 3 = 12$ $\_\_ + 3 = 12$.

b. Complete—declarative and having properties of being either true or false.

$10 + 8 = 18$

$5 - 6 > 9$

4. A common number sentence is often called a formula.

$A = lw$, $d = rt$

D. Variable introduced as an unspecified (or representative) element of some set of numbers (replacement set or domain)

The set of all x such that $x + 5 = 5$ where x is a positive integer—a symbol "x" used in this way is called a variable.

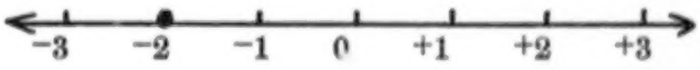
E. Solution sets of open sentences of one variable

1. Solution set may be thought of as the set consisting of all elements contained in a specified universe that satisfy the condition.

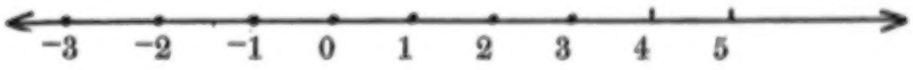
If the universe is $\{3, 4, 5, 6, 7\}$ then the solution set of $x + 2 = 6$ is $\{4\}$.

2. Graph of solution set of one variable

a. If the replacement set for x is the set of integers and $x + 5 = 3$, then the graph of the solution set of x is:

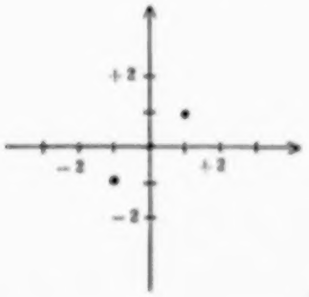


b. The graph of $\{x \in I \mid x + 4 < 8\}$ is:



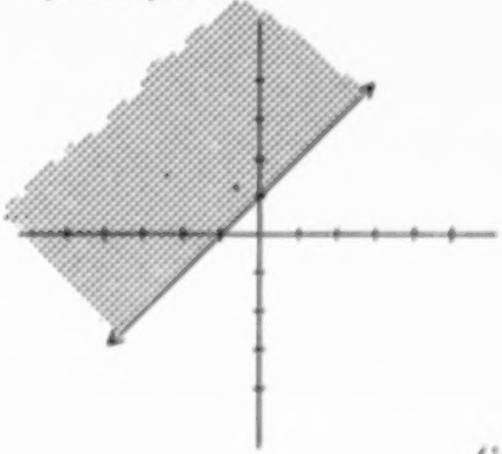
F. Solution set of two-variable sentences

1. Ordered pairs of real numbers
    - a. Number pairs (a,b) and (b,a), a and b are real numbers and $a \neq b$ are different pairs.
    - b. Definition of "=" for ordered pairs:
      For real numbers a, b, c, and d, $(a,b) = (c,d)$ if and only if $a = c$ and $b = d$.
2. One-to-one correspondence between the set of all pairs of real numbers and the points in the plane.
3. Solution sets of two-variable sentences as sets of ordered pairs of real numbers.
    - a. If the replacement set for x and y is $\{-1, 0, 1\}$ then the solution set of $x = y$ is $\{(-1, -1), (0,0), (1,1)\}$.
      Graph of $\{(-1,-1), (0,0), (1,1)\}$



(complete graph)

    - b. If the replacement set for x and y is the set of real numbers then the graph of the solution set of $y \geq x + 1$

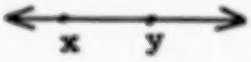
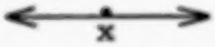
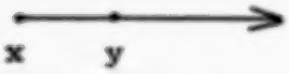
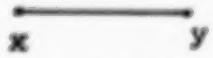
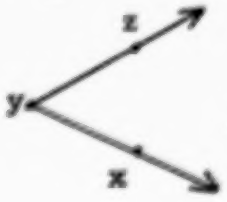
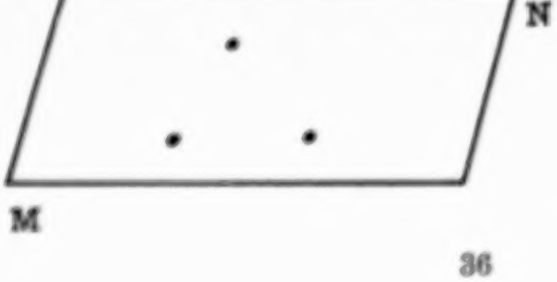


(incomplete graph)

G. Application—An abundance of meaningful exercises and problem situations should be presented with the development of concepts. Application involving one-variable sentences are usually found in adequate number in most texts. However, two-variable sentences, while rich and promising, are sometimes neglected. Some ingenuity may be called for. It should be kept in mind that most mathematics in the secondary schools could be taught from an order pair approach.

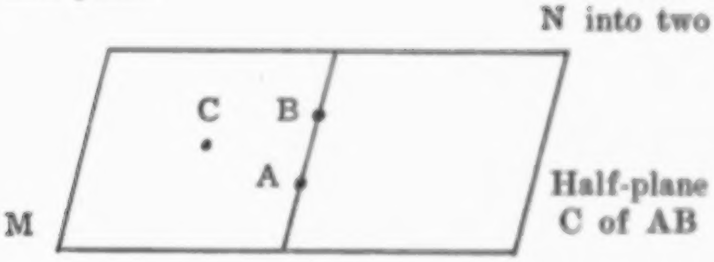
VII. Metric and Non-Metric Geometry

A. Basic properties of:

Set of Points	*Representation*	*Named*	*Property*
line		$\overleftrightarrow{xy}$	A line may be extended indefinitely in either direction.
half-line		$\circ\!\!\longrightarrow$	A point divides a line into two half-lines.
ray		$\overrightarrow{xy}$	A half-line and its endpoint.
segment		$\overline{xy}$	Two points determine one and only one line.
angle		∠XYZ or ∠Y	An angle is the union of two rays having a common endpoint or vertex.
plane		Plane MN	A plane is determined by (a) three points not in a straight line, (b) one line and one point not on the line, (c) two intersecting lines, and (d) two parallel lines.

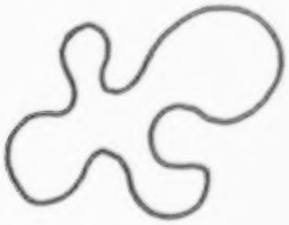
half-plane

A line divides a plane into two half-planes.



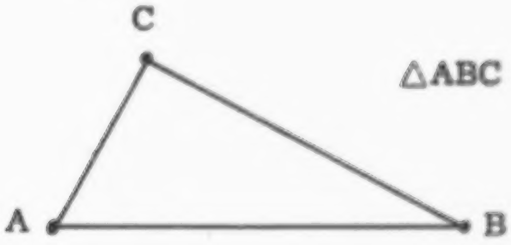
simple closed curve

A continuous line in a plane with no end points separates the plane into two regions—one *inside* with finite measure of area and one *outside*. It is also non-self-intersecting.



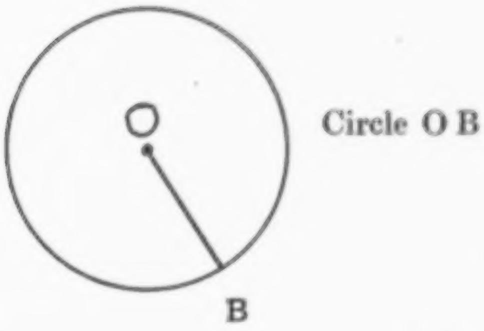
triangle

It is a simple closed curve which is the union of *three line segments*.



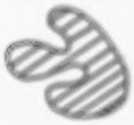
circle

A circle is a simple closed curve in a plane such that the points comprising this curve are a fixed distance from a point in the plane, called the center of the circle.

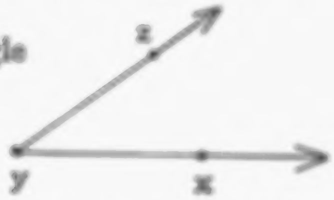


B. The nature of the measuring process

1. A convenient "unit" must be selected that is of the same nature as the thing being measured. If the unit is acceptable to a large number of people, it is called a standard unit.
   a. Line segment •————• to measure a line segment, use a line segment such as 0.001 inch, 1 foot, 1 mile, as a unit.
   b. Region enclosed within a simple closed curve. The unit of measure

   must be a region such as a square inch , a triangular inch , square mile, etc.
   c. Space bounded by surfaces—the unit must be a portion of space such as a cubic inch.
   d. Angle

   We use an appropriate angle such as a right angle

   or degree (angle)
2. Next, the thing being measured must be subdivided into portions which match (are congruent to) the unit.
3. The number of these subdivided portions is called the measure of the thing.
4. Instruments for measuring
   a. Line segment—ruler, yard stick, vernier caliper, steel tape, rolling wheel, etc.
   b. Region in a plane—plastic grid, planimeter.
   c. Space bounded by surfaces—by pouring liquid into graduated cylinder, by weighing object together with the unit of the same substance.
   d. Angle—protractor.
5. The approximate nature of the measuring process
   a. All measurements are approximate.

b. Precision is related to the size of the unit of measurement chosen—the smaller the unit, the more precise the measurement.

c. Accuracy is related to the relative error of the measurement. The smaller the relative error, the more accurate is the measure.

$$\text{Relative error} = \frac{\frac{1}{2}\text{ the smallest unit}}{\text{measure}}$$

d. The number of significant digits can express the accuracy of a number representing approximate data.

e. Scientific notation can be used to show the number of significant digits.

6. Skills in computing with approximate data:

a. Numbers may represent exact or approximate data. *Exact* data are obtained through counting discrete objects or through definitions. Example: There are 32 students in my class. There are 12 inches in a foot. Approximate data are obtained through measuring, rounding off numerals representing either exact or approximate data and computing with approximate data.

b. The sum or difference of numbers representing approximate data can be no more precise than the least *precise* of the numbers used.

c. The product or quotient of numbers representing approximate data can be no more accurate than the least accurate of the numbers used.

C. Construction

1. Copying line segments with compass. Bisecting line segments
2. Copying angles and bisecting angles
3. Constructing a line perpendicular to a line
4. Constructing a line parallel to a line
5. Constructing a triangle given one of these conditions:

        - a. 2 sides and included angle
        - b. 3 sides
        - c. 2 angles and included side
    6. Constructing an angle of 45° and also one of 30°
    7. Constructing regular polygons inscribed in a circle
    8. Constructing designs.

D. Similarity

1. Figures having the same shape are said to be similar.
2. Polygons are similar if their corresponding angles have the same measure and their corresponding sides are proportional, and conversely.
3. If 2 triangles have two pairs of corresponding angles with the same measures, then the 2 triangles are similar and their corresponding sides are proportional.
    - a. Use of similar triangles for indirect measurement
    - b. Tangent ratio
4. Scale drawings—using similar triangles and other similar polygons
5. (Optional) Any two plane (or 3-dimensional) figures are similar if for each pair of points A_1, B_1 in one figure there are corresponding points A_2 and B_2 in the second figure.

   $\frac{A_1B_1}{A_2B_2}$ = a constant.

   If the constant is 1, then the figures are congruent.

E. Congruence

1. Segment: Two line segments are congruent if they have the same measure.
2. Angles: Two angles are congruent if they have the same measure.
3. Triangles: Two triangles, ABC and DEF, are congruent if there is a correspondence ABC ↔ DEF so that one of the following conditions exists:

a. The corresponding sides are congruent (s.s.s.)

b. Two angles and the included side of one are congruent to the corresponding parts of the other (a.s.a.)

c. Two sides and the included angle of one are congruent to the coresponding parts of the others (s.a.s.).

F. Parallelism

1. Lines

a. Two lines which are in the same plane and never meet are parallel.

b. Parallel lines have the same slope, ratio of rise to run.

c. A straight line is parallel to a plane if it is at all times equidistant from the plane.

d. If two lines are parallel, every plane containing one of the lines, and only one, is parallel to the other.

e. If two lines are parallel to a third line, they are parallel to each other.

f. When two parallel lines are cut by a transversal, the corresponding angles are equal.

g. When two parallel lines are cut by a transversal, the alternate interior angles are equal.

h. Two lines perpendicular to a third line all in the same plane are parallel.

i. If a line joins the middle points of two sides of a triangle, it is parallel to the third side and equal to one half of it.

2. Planes

a. If two lines are cut by three parallel planes, the corresponding segments are proportional.

b. Through a given point X, outside a given plane R, there is only one plane which can be parallel to R through point X.

c. Two planes perpendicular to the same straight line are parallel.

d. If two parallel planes are cut by a third plane, the lines of intersection are parallel.

G. Locus

Defined as the set of points and only those points that satisfy a certain given condition. Contrast locus on a plane and locus in space.

Examples: The Locus of Points

1. In a plane equally distant from 2 fixed points
2. In space 6" from a fixed point
3. In a plane equally distant from the intersecting x-and y-axes
4. In space equally distant from the 3 surfaces (floor, north wall, east wall) of a classroom
5. In the plane of a triangle equally distant from the 3 vertices of the triangle.

VIII. Irrational Numbers

A. Interpretation

1. In presenting some of the introductory work with the irrational numbers, it is assumed that they will be considered more from a useful and problem-solving situation than from a formal, definition approach.
Introduction of the irrationals will eventually, in algebra, lead to the completion of the *set of real numbers*, which is the union of the set of rationals and the set of irrationals.

These numbers are needed to express mathematically the meaning of concepts, such as length, area, and volume, thus their introduction can be presented on that basis.

*2. Formally, the irrationals are a sub-set of the *real numbers* where the members of the set of real numbers are thought of as the *least upper bound* of a set of rational numbers. For instance, 3 can be thought of as the least upper bound of the set of all rational numbers of the form $3 - \frac{1}{n}$, where n is a positive integer. The set will look like this:

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* Advanced topic

$$\{2, \frac{5}{2}, \frac{8}{3}, \frac{11}{4}, \frac{14}{5} \ldots\}$$

B. Square Roots

1. a. If a number is a product of two identical factors, either factor is called a square root of the element or number. The symbol "$\sqrt{\ }$" is used to designate the positive square root of a number.

1. b. In the set of numbers of arithmetic, each member has either one or no square root.

$$\sqrt{\frac{9}{16}} = \frac{3}{4} \text{ since } \frac{9}{16} = \frac{3}{4} \times \frac{3}{4}$$

$\sqrt{6}$ is meaningless in the numbers of arithmetic.

1. c. As the set of integers or rationals are introduced, some numbers will now have two square roots.

Since $25 = (^{-}5)\ \ (^{-}5)$ and
$25 = (5)\ \ (5)$
25 has two square roots.
$\sqrt{25}$ means 5 and $^{-}\sqrt{25}$ means $^{-}5$.

2. Determination of square root (approximately)
   a. By a table
   b. Computation by the "divide and take the average method"[3]
   *c. Computation by the "multiply by 20" method.

C. The Pythagorean Relationship

1. The statement can be informal and would emphasize its immediate use to triangles. Its use develops a use for squares and square roots.

*2. Intuitive proofs can be presented or a simple deductive one using the area of a square region.

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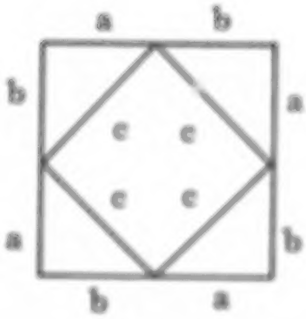
[3] Peterson, John and Hashisaki, Joseph. *Theory of Arithmetic*. New York: John Wiley, 1963, p. 223f.

* Advanced topic



Area of entire square is $(a + b)^2 = a^2 + 2ab + b^2$
also it $= 4(\frac{1}{2}ab) + c^2$
thus $a^2 + 2ab + b^2 = 4(\frac{1}{2}ab) + c^2$
or $a^2 + b^2 = c^2$

3. Application to problems related to life situations.

D. Pi (π)

1. Meaning and interpretation: to relate the meaning of pi at the beginning stages, many circular objects should be brought into the room and their diameters, radii, and circumferences measured; using a table of these measurements, the meaning of pi can be more easily developed. For example, use a table like this:

Object	c	r	d	$c - d$	$c - r$	$c \times d$	$c \times r$	$c \div d$	$c \div r$
I									
II									
III									

2. Use of pi with areas of circular regions. Include measuring the area as well as computing with the formula. The area can be measured by placing a sheet of clear plastic with a grid of square regions over the region to be measured and then counting the squares which are entirely or over half way within the circle. A logical development for the formula should be given, too, before used extensively.

3. Approximate values for *Pi:*
Use the number line to serve as a visual interpretation for selecting rational numbers with the desired accuracy (number of places). Possible approximations are: 3 (nearest integer), $3\frac{1}{7}$, 3.14, and 3.1416.

4. Application to problems related to life situations.

E. Decimal Approximations for Irrationals

  1. Irrationals can be interpreted as non-repeating, nonterminating decimals.
  2. Association of the irrationals with the points on a number line along with the rationals (to form the real number line). This association permits the selection of a rational number to any desired degree of accuracy.

IX. Statistics and Probability

A. Meaning

  1. *Statistics* is the mathematical science that deals with the collection, organization, and description of numerical data.
  2. *Probability* is the mathematical science that deals with predicting the likelihood of events happening.

B. Obtaining data

  1. Data are obtained through *counting*, *measuring*, and *computing*.
  2. Data should be as accurate and reliable as possible.

C. Statistical measures which may be used to aid one to answer questions.

  1. What single number is representative of a set of data?
     a. Mode
     b. Median
     c. Mean
  2. What is the relative standing of some particular number within a set of data?
     a. Rank
     b. Percentile rank
  3. How much scatter or variability is there within the data?
     a. Frequency table
     b. Range

  *4. If one has two sets of data for a set of objects, how are these two sets of data related? A scatter diagram is quite useful here and is not difficult for students to understand. For example, suppose you obtain the measures of the forearm and size of shoes worn by the members of the class. Using a coordinate graph like the one shown here, one can place a dot for each student.

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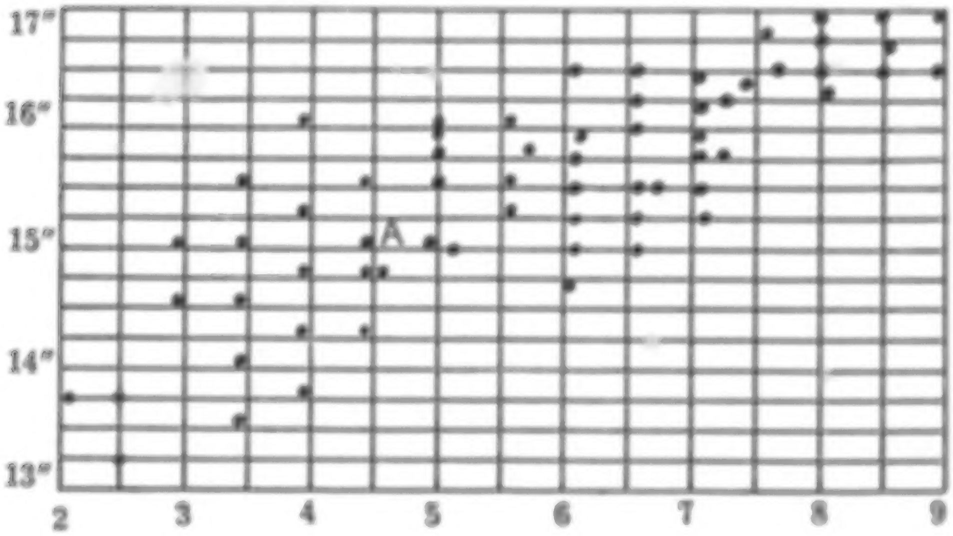
* Advanced topic



From this scatter diagram one could answer such questions as the following:

a. Look at point A. This point represents the shoe size and forearm length of some one student. What is his (or her) shoe size? Length of forearm?

b. Do any other students wear the same size shoe as the student represented by point A?

c. Do students who wear size 7 shoes all have the same length of forearm?

d. Compute the mean of the lengths of forearm for the 3 students wearing size 7 shoes.

e. How does this mean compare with those wearing size 4 shoes?

f. Place a pencil in a vertical position so that half of the points are on the right and half on the left of the pencil. How does this enable you to determine the median shoe size? What is the median shoe size? What is the median arm length?

g. What is the largest shoe size shown? The smallest? The range of shoe sizes?

h. Is there a tendency for people with the largest shoe size to have the longer arm?

D. Graphic methods for presenting numerical data

1. Pictograph
2. Bar graph
3. Circular graphs
4. Broken line
5. Rectangular distribution
6. Histogram.

*E. Probability (Optional)

1. Definition—Probability is a measure of the likelihood or chance of event happening. Probability is written as a ratio of the number of ways an event can happen successfully to the total number of ways an event can occur (both successfully and unsuccessfully).

2. Probability may deal with events

   a. in which there is equal likelihood assumed for each way an event can happen

   b. in which the likelihood for an event happening is unknown and must be approximated from samples (empirical probability).

3. Probability of equal likelihood events. In determining the number of ways an event can happen, we will introduce the concepts of arrangements (that is, permutations) and selections (that is, combinations) of distinguishable objects. Tree diagrams and rectangular arrays may be used to show that two independent events can happen in $m \cdot n$ different ways if one event can happen in m ways and if for each of these ways a second event can happen in n ways. This principle leads to the development of formulas for the number of ways a collection of n distinguishable objects can be arranged.

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* Advanced topic

a. n distinguishable objects can be placed in n! different arrangements having the *n* objects in each arrangement.

$${}_nP_n = n!$$

b. n distinguishable objects can be placed in $\frac{n!}{(n-r)!}$ different arrangements each having r objects. ${}_nP_r = \frac{n!}{(n-4)!}$

c. One man make $\frac{n!}{r!\,(n-r)!}$ different selections of r objects from a collection of n objects, the order of arrangement of the objects within each selection not being considered.

$${}_nC_r = \frac{n!}{r!\,(n-r)!}$$

4. Empirical probability. Examples:
   a. Mortality tables
   b. Sampling technique to determine the probability that a ball drawn from a box of red and black balls will be black, the number of balls of each color in the box being unknown.

*X. Number Expression (Polynomials)

An extensive development of polynomials should be delayed until algebra. The point of view taken here is introductory and limited in breadth and depth. Definitions should be descriptive. Many examples should be utilized. Emphasis should be placed on application wherever feasible, such as scientific notation related to space mathematics and the laws of exponents.

A. A polynomial is a name for a number. If the polynomial contains a variable such as $3x^2 - x$, the number being named depends upon the replacement of x. Examples of polynomials: 17, $8y - 2$, $3xy + y^2 + 5$.

B. Standard ways of writing polynomials—ascending or descending order.

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* Advanced topic

C. Terms of a polynomial

  1. Monomial
  2. Binomial
  3. Trinomial

D. Degree of a polynomial

E. Operation with numbers in polynomial form containing variables

  1. Addition—combining similar terms
  2. Subtraction—additive inverse
  3. Distributive principle should be utilized. $3x^2 + 5x^2 = (3 + 5)x^2 = 8x^2$
  4. Multiplication
     a. Exponential law: $a^n \cdot a^m = a^{n+m}$ where m and n are natural numbers
     b. Number in monomial form times another in monomial form
     c. Number in monomial form times another in binomial form
     d. Number in monomial form times another in polynomial form
     e. Number in binomial form times another in binomial form.
  5. Division
     a. Exponential law for division: $a^n \div a^m = a^{n-m}$ where m and n are natural numbers and $n > m$.
     b. Divisors should be limited to monomials; examples should be of the kind that do not require working with negative exponents, such as:

$$\frac{6x^2}{3x} \qquad \frac{3x^2 + 9}{3} \qquad \frac{5x^2 + 4x^3y^4}{x^2y}$$

G. Evaluating polynomials

Example: Evaluate $3x^2 + 4y$ if $x = 3$ and $y = 5$

$$3(3)^2 + 4(5) = 27 + 20 = 47$$

2

*XI. Concept of Proof and Logical Structure

A. Discovery of patterns among data. Develop the idea that mathematics is concerned with patterns. Many of these patterns can be discovered and making such discoveries can be an exciting experience. In order to facilitate student discovery of patterns, the data should be well organized and students should be encouraged to make conjectures and to predict by extending the pattern to new data. They can receive a thrill from making a prediction and having that prediction verified. For example, consider the data below:

$$\begin{aligned}
1 &= 1 = 1^2\\
1 + 3 &= 4 = 2^2\\
1 + 3 + 5 &= 9 = 3^2\\
1 + 3 + 5 + 7 &= 16 = 4^2\\
1 + 3 + 5 + 7 + 9 &= 25 = 5^2
\end{aligned}$$

What patterns do you notice in the addends on the left? What is the sum $1 + 3 + 5 + 7 + 9 + 11$? Can you predict that sum $1 + 3 + 5 + \ldots + 17$? How many addends? How many addends in the following $1 + 3 + 5 + \ldots + 53$? Predict the sum.

Discovery exercises are appropriate throughout much of junior high school mathematics. Students may suggest ways of extending the patterns observed; however, such generalizations must be treated as conjectures. The following are a few of such discovery activities:

1. Relationship between measure of diameter and the circumference of circles.
2. If two triangles have two pairs of corresponding angles with the same measure, then the measures of their corresponding sides are proportional.
3. How many diagonals can be drawn in a polygon of 5 sides? 6 sides? n sides?
4. The number of subsets of a set of n elements $= 2^n$.
5. The difference between squares of successive whole numbers.

° Advanced topic

B. *Truth table for compound statements.*

p	q	$p \wedge q$	$p \vee q$
T	T	T	T
T	F	F	T
F	T	F	T
F	F	F	F

C. *The "If-Then" Relationship*—Some conditions can be observed as the consequences of other conditions.

1. If 3 sticks (line segments) have measures in the ratio of 3 : 4 : 5, then the triangle formed from these sticks appears to always be a right triangle.
2. If a number N is a multiple of 9, then the sum of its digits (in base ten numerals) is also a multiple of 9.
3. What is the meaning of an if-then statement?
4. (Optional). Converse and contrapositive forms of if-then statements

If p . . . then q.	Statement
If q . . . then p.	Converse form which may or may not be true

when the original statement is true.
If not q . . . , then not p. Contrapositive form which is equivalent to the original statement.

5. Truth table for "If . . . , then" statements.

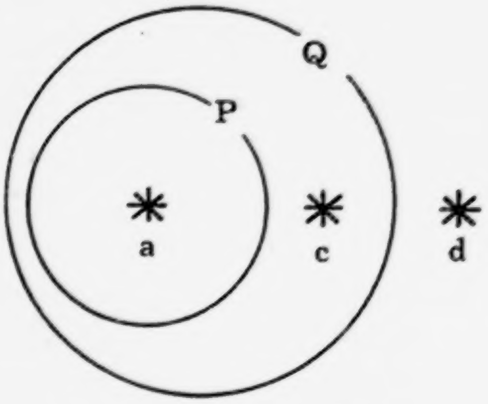
D. *Euler Diagrams* (Optional)

1. To illustrate "If p, then q" statements:
Such a diagram must provide 3 regions that correspond to the cases a, c, and d in the truth table. If "If p, then q" is a true statement, it is impossible for one to find an instance in which p is true and q is false; consequently

there can be no region in the diagram in which a point can be in circle P and outside of circle Q.

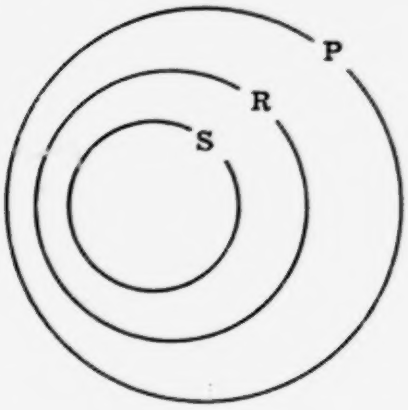
	p	q	If p , then q
a	T	T	T
b	T	F	F
c	F	T	T
d	F	F	T

Euler Diagram for "if p, then q."



There is no place for a point to represent case b —in P and outside of Q.

2. Euler diagrams can be used effectively to show the relationship between sets of objects such as:
   a. Set of all squares (S), set of all rectangles (R), and set of all parallelograms (P).



   - b. Set of all multiples of 5(A), set of all odd numbers (B), and set of all primes (C).

E. Introduction to Deductive Proof

Students in the junior high school years can be introduced to deductive type of proof.

1. In mathematics classes in the elementary school they begin to give reasons to support their decisions when they are faced with the question "why?" It should become more apparent to students in the junior high school that they are asked to accept certain ideas about mathematics and that the various manipulative steps are the consequence of these ideas. For example, from the properties of whole numbers one can develop algorisms or techniques for computation using multidigit numbers.
2. Definitions are very important if one is to work with mathematical ideas. If one is working with odd numbers, how can we define an odd number?
3. From accepted mathematical ideas one can develop new ideas.
   - a. Using the fact that the sum of the measures of the angles of a triangle is 180°, one can deduce the sum for a quadrilateral is 360°.
   - b. From the properties of whole numbers and, of course, the definition of an odd number, one can prove that the set of odd numbers is closed with respect to multiplication.
   - c. From the postulate that the area of rectangle is equal to the product of the length by the width, one can develop a method for computing the area of other geometric figures.

F. (Optional). Introduction to the idea of a mathematical system having undefined terms, postulates, definitions, and deducible theorems.

1. Modulo arithmetic
2. Permutation groups.

CHAPTER 2

Algebra

THE MATHEMATICS CURRICULUM most commonly used today was formulated, for the most part, about thirty years ago. In the meantime, the traditional courses in elementary and intermediate algebra have become established primarily as a collection of manipulative skills. The usual method of presentation has been for the teacher to present a rule, work a few examples, and assign similar examples for students to work. This procedure has naturally led many students to consider algebra as a collection of "tricks" with very few relationships among the rules or with the other areas in mathematics.

The modern point of view considers algebra as the study of mathematical structure in which the student is helped to discover most of the subject matter for himself and to look constantly for unifying ideas that apply in algebra as well as in all mathematics. Practically all manipulations are performed only after a reasonable understanding has been developed by students.

The particular objective of this course is to present algebra according to the best contemporary point of view. The committee believes that the following outline can be developed into courses that emphasize this point of view. In the proposed outline, the formal manipulations in algebra are the same as those taught in the past, and the subject matter is largely the same with the exception of inequalities, which are treated both algebraically and graphically. The main difference is in concept, terminology, and symbolism. This new emphasis upon the understanding of the fundamental ideas is not to imply that strong skills are not needed, for they most surely are, but they must be based on understanding and not merely on rote memorization.

The topics in this section of the guide are normally thought of as belonging to Algebra I and Algebra II. Some additional algebra is included in the Advanced Mathematics section. No attempt has been made, however, to specify firmly minimum or maximum requirements for Algebra I or Algebra II. The committee strongly believes that each school or county system should set its own course requirements in order to adjust to its faculty and students. Thus the algebra of this section may be covered in more or in less than two years.

As you read the outline, please keep the following points in mind:

1. The sequence of topics is subject to change.
2. Part I and most of Part II should be introduced in junior high school, formalized in Algebra I, and reviewed in Algebra II.
3. Topics without asterisks are mainly a part of Algebra I; topics with asterisks are mainly a part of Algebra II.

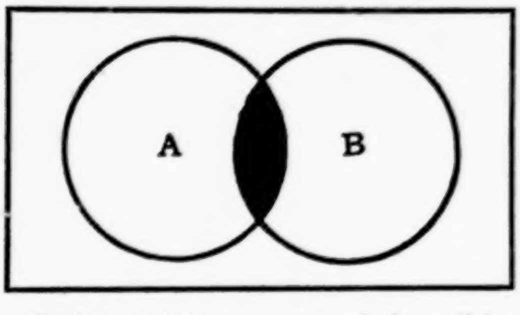
Algebra Outline

I. Language and Symbolism of Numbers

Language of Sets

Define and give symbols for such terms as universe; subsets, both proper and improper; elements of a set; membership (ϵ); null set or empty set; union; intersection; and complement. Use Venn diagrams to aid definitions. For example:

Intersection of two sets A and B is the set containing everything that belongs only to A and B. Symbol: $A \cap B$



The shaded portion of the diagram illustrates the intersection of sets A and B.

Point out two ways of describing sets:

Roster method. Example: $\{0, 2, 4, 6, 8, \ldots\}$

Rule method. Example: {positive even integers}

Example: $\{X \mid X \epsilon R, X > 5\}$

Discuss and give examples of two kinds of sets—finite and infinite.

II. Real Numbers

(This is a review of number systems previously studied; some from a more sophisticated point of view. For a more comprehensive review read sections I, III, IV, V, VI, VIII, X of Chapter 1.)

A. Properties of the numbers of arithmetic (whole numbers and fractions). Illustrate and state:

1. Closure under addition, multiplication, and division but not subtraction
2. Commutative and associative laws for addition and multiplication, but not for subtraction and division
3. Distributive law of multiplication with respect to addition—(used to simplify expressions — to "combine terms")
4. Order—*i.e.*, real numbers can be ordered on the number line.

Define identity and inverse elements.

Special properties of the numbers 1 and 0

Additive identity: There exists an element 0 such that for every a, $a + 0 = 0 + a = a$

Multiplicative identity: There exists an element 1 such that for every a, $1 \cdot a = a \cdot 1 = a$

Multiplicative property of 0: For every a, $a \cdot 0 = 0 \cdot a = 0$

As a result of this property, it is true that $\frac{0}{a} = 0$ for every $a \neq 0$. According to this property $\frac{0}{0}$ is indeterminate and $\frac{a}{0}$ is meaningless. ($\frac{a}{0} = \infty$ is improper notation)

Multiplicative inverses—but not additive inverses until ready to introduce directed numbers.

Properties of equality: $a = a$ (reflexive)
If $a = b$, then $b = a$ (symmetric)
If $a = b$ and $b = c$, then $a = c$ (transitive)

Inverse operations: addition and subtraction
multiplication and division
raising to a power and extracting roots

B. Symbols of Grouping

Order of operation—

Purpose of symbols is to prevent ambiguous expressions such as $4 + 5 \times 3$. Is it 27 or 19? Hence the use of parentheses ( ), brackets [ ] and braces $\{\}$ make the meaning of such expressions clear. It is agreed that one should perform the operations inside the symbol of grouping first. Frequently the symbols of grouping are omitted and hence mathematicians have agreed on a rule. That is, whenever more than one operation occurs in a problem, the multiplication and division is performed in order from left to right and then the addition and subtraction in order from left to right.

Thus, $6 \times 7 - 4 \times 3$ means $(6 \times 7) - (4 \times 3) = 30$ and $40 \div 8 \times 2$ means $(40 \div 8) \times 2 = 10$.

C. Integers

Arrive at integers by use of a number line. The following points should be brought out concerning the number line:

1. One-to-one correspondence between a subset or points on a line and integers. (Point out that + and − signs now have two distinct meanings—operation and direction. Suggest using raised symbols for integers so as to distinguish between operation sign and direction sign.)
2. Integers are ordered.
3. Every integer has an opposite or inverse. (The opposite of a positive number is a negative number; the opposite of a negative number is a positive number; and the opposite of zero is zero.)
4. Addition of integers; use of number line. (It is best to avoid a formal rule yet.)
5. Absolute value: (suggested for developing rule)—often it is necessary to work with the magnitude, not the direction, of an integer. For example, a measurement on the number line from -5 to $+2$ is 7. Conversely, from $+2$ to -5 is 7.

a. Descriptive definition— $|5| = 5$
$|-5| = 5$
$|0| = 0$

b. Explicit definition—For all x in the set of integers,
if $x \geq 0$, then $|x| = x$
if $x < 0$, then $|x| = -x$

6. Addition of integers:
   a. If a and b are each positive or zero, then $a + b = |a| + |b|$
   b. If a and b are each negative or zero, then $a + b = -(|a| + |b|)$
   c. If a is positive and b is negative and $|a| \geq |b|$, then $a + b = |a| - |b|$
   d. If a is positive and b is negative and $|a| < |b|$, then $a + b = -(|b| - |a|)$
   e. If a is negative and b is positive and $|a| > |b|$, then $a + b = -(|a| - |b|)$
   f. If a is negative and b is positive and $|b| > |a|$, then $a + b = |b| - |a|$

7. Subtraction of integers—defined as opposite or inverse operation of addition; that is, $a - b$ means $a + (^{-}b)$.

8. Multiplication of integers (several choices):

   a. Introduced through number line. For example, to multiply $(^{+}5)\ (^{-}2)$, use idea of directed rate and directed time. Thus, to determine the value of $(^{+}5)\ (^{-}2)$ think of going eastward (positive direction) at the rate of 5 m.p.h. Interpret $^{-}2$ as “2 hours ago.” Ask, “Where were you 2 hours ago?” (10 miles in the opposite direction [west of where you are now].) To illustrate $(^{-}5)\ (^{+}2)$, think of going westward (negative direction) at the rate of 5 m.p.h. Where will you be 2 hours from now? or

   b. Approached by preserving the distributive law. Example: Since $7 + (^{-}7) = 0$ and $8 \cdot 0 = 0$ then
   $8\ [7 + (^{-}7)] = 0$
   By distributive law, $8\ [7 + (^{-}7)] = 8(7) + 8(^{-}7)$
   So $8(7) + 8(^{-}7) = 0$

Since $8(7) = 56$, then $8(^-7) = ^-56$ since additive inverse is unique.

Hence, $^-8\,[7 + (^-7)] = 0$ and $^-8(7) + (^-8)\ (^-7) = 0$

Since $^-8(7) = -56$, then $(^-8)\ (^-7) = {}^+56$ since additive inverse is unique.

c. "Continuity argument" (see *Modern Elementary Algebra* by E. D. Nichols).

9. Division of integers—defined as inverse operation of multiplication. Thus, $^+15 \div {}^-3$ means $^-3 \times ? = {}^+15$.

10. Properties of integers: (summary)

    a. Closure under addition, subtraction, multiplication, but not division.

    b. Addition and multiplication are commutative.

    c. Addition and multiplication are associative.

    d. Multiplication is distributive with respect to addition.

    e. Every integer has an additive inverse, but not a multiplicative inverse.

    f. Zero is the additive identity. One is the multiplicative identity.

D. Rationals

1. Definition

2. Properties

    a. Since the set of rationals includes the set of integers, all of the properties of integers hold for the rationals.

    b. Every non-zero rational has a multiplicative inverse.

    c. Density (between every two rationals there is another rational).

3. Decimal representation
Every rational number can be represented as a repeating or terminating decimal and conversely.

E. Irrationals:

1. Definition
2. Properties
    - a. Non-repeating decimal representation
    - b. Point out no closure under addition or multiplication For example,

$$\begin{array}{r} 0.342342234222\ldots \\ +0.657657765777\ldots \\ \hline 0.999999999999\ldots \end{array}$$

which is the rational number 1 or $\sqrt{3} \cdot \sqrt{12} = \sqrt{36} = 6$

Note: Union of the set of rationals with the set of irrationals is called the set of reals.

*F. Properties of groups and fields (See Allendoerfer and Oakley's *Principles of Mathematics.*)

Note: Informal discussion in Algebra I; more in Algebra II.

III. Equations and Inequalities in One Variable

A. Linear Equations

1. Define such terms as variable, term, factor, expression, value of expression, equivalent expressions (use of quantifiers), equation, solution set, root, and equivalent equations. For example, define equation as follows:

An equation is any mathematical sentence using the symbol =. Equations can be classified into two kinds of sentences—closed and open. A closed sentence may be either true or false. For example,

$2 + 3 = 5$ which is true

$2 + 3 = 6$ which is false.

An open sentence has three possibilities. For example,

$X^2 = 9$ is true of some but not all replacements of X.

$X + 1 = X + 2$ is true of no replacement of X.

$X + 2 = 2 + X$ is true of each replacement of X.

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* Topics with asterisks are mainly a part of Algebra II; topics without asterisks are mainly a part of Algebra I.

2. Intuitive approach (To be covered only if not done previously) Solving simple equations by having them guess the roots—For example, $N + 7 = 12$, what number added to 7 gives 12? or another example,

   $\frac{2X - 5}{3} = 5$. Cover the numerator and ask what number divided by 3 gives 5. So $2X - 5 = 15$; hence, what number minus 5 gives 15? Therefore, 2 times what number gives 20?

3. Axiomatic approach
   Equivalent equations and what transformations lead to equivalent equations.

   Solving equations which contain variables on both sides motivates the need for transformation principles.

B. Absolute value equations

Example: $|2X - 5| = 7$ by definition of absolute value

$2X - 5 = 7$ or $2X - 5 = {}^-7$

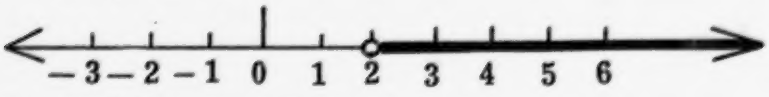
Hence, the solution set is $\{6, {}^-1\}$.

C. Inequalities

1. Definitions such as inequality, solution set, etc.

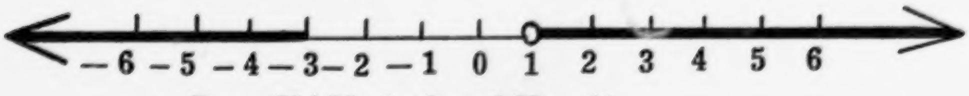
2. Graphical representation on the real number line. Consider the following examples:

   a. $S = \{X \mid X > 2\}$

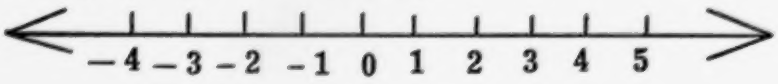


   b. $T = \{X \mid X \leq {}^-3 \text{ or } X > 1\}$

   (${}^-3$ is included; 1 is excluded)



   c. $R = \{X \mid X \leq {}^-3 \text{ and } X > 1\}$



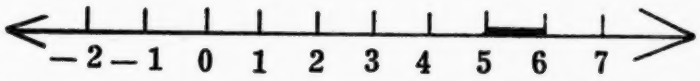
Since this graph is the intersection of disjoint sets, then

$R = \{X \mid X \leq {}^-3 \text{ and } X > 1\} = \phi$

This can also be written as

$R = \{X \mid X \leq {}^-3\} \cap \{X \mid X > 1\} = \phi$

d. $K = \{X \mid X \geq 5 \text{ and } X \leq 6\}$

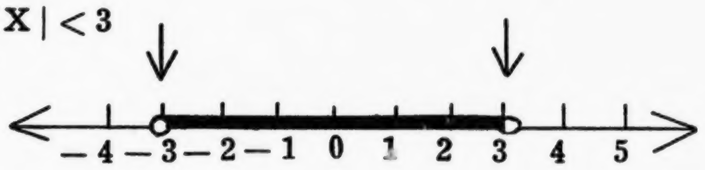


Hence $K =$ {numbers between and including 5 and 6}

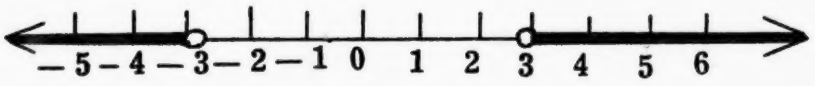
or $K = \{X \mid X \geq 5\} \cap \{X \mid X \leq 6\} = \{X \mid 5 \leq X \leq 6\}$.

e. Absolute value inequalities—solved graphically

(1) $|X| < 3$



(2) $|X| > 3$



f. It is not necessary to use the set of real numbers as the universe—one may choose the set of integers (or naturals).

$L = \{X \mid X \in \text{Integers}, {}^-3 < X < 2\}$

−5 −4 −3 −2 −1 0 1 2 3 4

(Note that for integers, the "number line" consists of points which are not "connected".)

3. Properties

a. Order—for each a and b, one and only one of the following is true: $a = b$, $a<b$, or $a>b$. (Called the *law of trichotomy*)

b. For any a, b, and c

(1) if $a < b$ and $b < c$, then $a < c$.

(2) if $a > b$ and $b > c$, then $a > c$.

(3) if $a < b$, then $a + c < b + c$.

(4) if $a > b$, then $a + c > b + c$.

(5) if $a < b$, then $ac < bc$ if $c > 0$
$ac = bc$ if $c = 0$
$ac > bc$ if $c < 0$

(6) and inversely, if $a < b$, then $\frac{a}{c} < \frac{b}{c}$ if $c > 0$
$\frac{a}{c} > \frac{b}{c}$ if $c < 0$

4. Solving inequalities using the above properties (include absolute value inequalities).

D. Verbal problems should be spread throughout every year's course.

IV. Polynomials (introduce by examples before defining)

A. Definitions—polynomial, numerical coefficient, number of terms, degree

B. Universe for coefficients and range of variables

C. Operations with polynomials—(need some of the laws of exponents)

D. Evaluation of polynomials

E. Factoring—distributive principle (Factors are usually polynomials with coefficients from rational field)

1. Removing a common factor
2. Difference between two squares
3. Perfect square trinomial
4. Trial and error factoring of the general quadratic

F. Simplification of rational expressions (addition, subtraction, multiplication, and divison)

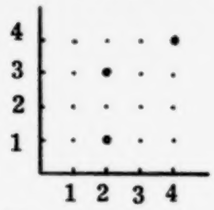
G. Simplification of complex fractions

(Note: This part should be reviewed in Algebra II.)

V. Relations and Functions

A. Relations

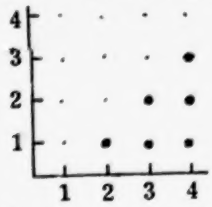
1. Define ordered pairs and Cartesian product. Then, by a relation we mean any set of ordered pairs. By a relation in U, we mean a subset of $U \times U$. The set of the first coordinates of a relation is called the domain of the relation. The set of second coordinates of a relation is called the range of the relation. Thus, in the set $A = \{(4,1), (6,7)\}$ the domain is $\{4,6\}$ and the range is $\{1,7\}$.
2. Graphs of relations:
   a. $R_1 = \{(2,3), (4,4), (2,1)\}$



Domain: $\{2,4\}$

Range: $\{1,3,4\}$

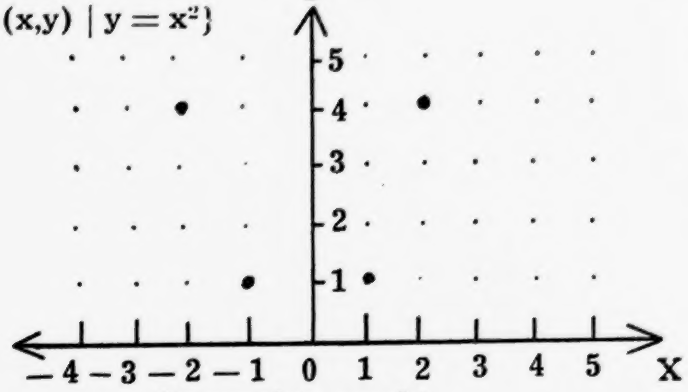
   b. $U = \{1,2,3,4\}$
   $R_2 = \{(x,y) \mid x > y\}$



Domain: $\{2,3,4\}$

Range: $\{1,2,3\}$

   c. U = set of integers
   $R_3 = \{(x,y) \mid y = x^2\}$

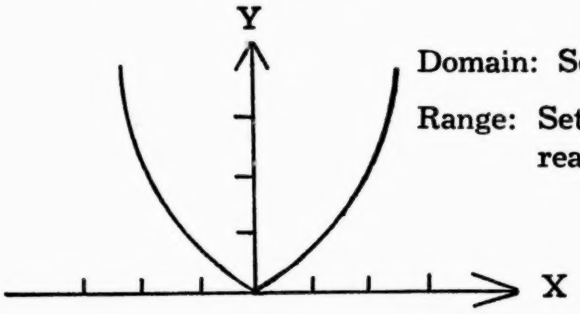


(incomplete graph)

Domain: Set of integers
Range: $\{y \mid y$ is the square of an integer$\}$.
$2 \notin$ range

d. U = set of reals

$R_4 = \{(x,y) \mid y = x^2\}$



Domain: Set of reals

Range: Set of non-negative reals

(incomplete graph)

3. Inverse and graph of inverse

   a. Inverse of a relation is the relation formed by interchanging the first and second coordinates in each ordered pair.

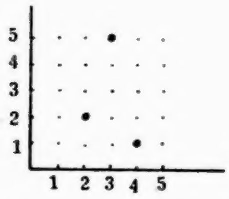
      Example: $S = \{(2,1), (3,2), (4,1)\}$

      Inverse of $S = \{(1,2), (2,3), (1,4)\}$

      (Inverse of S is often denoted by S' or by S^{-1}.)

   b. Graph of relation and its inverse

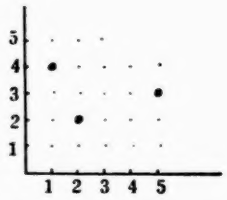
      $R = \{(2,2), (3,5), (4,1)\}$



Domain: {2,3,4}

Range: {1,2,5}

Inverse of $R = \{(2,2), (5,3), (1,4)\}$



Domain: {1,2,5}

Range: {2,3,4}

(Note that each of these graphs is a reflection of the other in the line graph of $y = x$.)

B. Functions

1. Definition—A function in U is a relation in U having for each x at most one y
2. Ways of describing—roster, rule, table, and graph
3. Vertical line test

   Since the first components of the ordered pairs are usually associated with the horizontal axis and the second components with the vertical axis, then we say that a relation is a function, if and only if, in the graph of the relation, no vertical line meets the graph of the relation at more than one point.
4. Graphs on lattices (finite and then the real number plane)
5. Notation and evaluation—distinguish among x, f(x), (x,f(x)), and f

   Composition of functions
6. Inverse

   Example: $R_1 = \{(2,3), (5,3)\}$; R_1 is a function

   Inverse of $R_1 = \{(3,2), (3,5)\}$; not a function

   Example: $R_2 = \{(3,7), (2,5)\}$, a function

   Inverse of $R_2 = \{(7,3), (5,2)\}$, a function
7. Graph of inverse
8. Cases of discontinuity

   f: $f(x) = \dfrac{1}{1-x}$ g: $g(x) = \sqrt{x-3}$

   (Note: This part should be introduced in Algebra I and extended in Algebra II.)

VI. Systems of Linear Equations and Inequalities

A. Equations—two variables

1. Graphing a linear function ($y = mx + b, m \neq 0$)
   Definitions such as x-intercept, y-intercept and slope (vertical line has no slope; horizontal line has slope of zero)

2. Solving systems of equations graphically
   Three cases—intersecting lines; parallel lines; coinciding lines

3. Solving systems of equations algebraically
   a. Elimination by addition and subtraction
   b. Elimination by substitution
   c. General solution (determinants)

4. Verbal problems

*B. Equations—three or more variables
algebraic solution

C. Inequalities in two variables

1. Solving an equation and an inequality both graphically and algebraically.

2. Solving two or more inequalities both graphically and algebraically.

VII. Exponents and Radicals

A. Exponents

1. Definitions

2. Laws of positive integral exponents

3. Rational exponents (positive, negative, and zero—defined to preserve laws of positive exponents)

B. Radicals

1. Definitions and symbolism (stress $\sqrt{a} > 0$ for all $a \geq 0$ and $\sqrt{a^2} = \left\{\begin{matrix} a, a \geq 0 \\ -a, a < 0 \end{matrix}\right\}$

2. Operations with radicals (stress $\sqrt{a} \cdot \sqrt{b} = \sqrt{ab}$ if $a \geq 0$, $b \geq 0$ but if $a < 0$ and $b < 0$ then $\sqrt{a} \cdot \sqrt{b} \neq \sqrt{ab}$)

*3. Solving equations involving radicals (stress operations which may not lead to equivalent equations.)

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* Topics with asterisks are mainly a part of Algebra II; topics without asterisks are mainly a part of Algebra I.

C. Scientific notation
  1. Definition
  2. Examples and problems
  3. Order of magnitude, for example, is 10^{12} nearer to 10^{10} or to 10^{13}?

VIII. Quadratic Equations, Functions, and Inequalities in One Variable

A. Quadratic equations
  1. General form $(ax^2 + bx + c = 0, a \neq 0)$
  2. Quadratic equations with rational roots

     Solved by factoring (based on principle: if a and b are any two real numbers such that $ab = 0$, then $a = 0$ or $b = 0$ or both a and b equal zero).
  3. Quadratic equations with real roots
    a. Solved by completing the square
    b. General solution (formula)
  *4. Sum and product of roots
  *5. Equations of higher degree which are reducible to quadratic form
  6. Verbal problems

*B. Quadratic functions
  1. Definition
  2. Graph
    a. Parabola—general form $y = a(x - k)^2 + p$

       (Show the effects of a, k, p.)
    b. Circle, ellipse, hyperbolas
      (1) general forms
      (2) curve-sketching

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* Topics with asterisks are mainly a part of Algebra II; topics without asterisks are mainly a part of Algebra I.

    3. Discuss vertex, maximum and minimum values, axis, symmetry, opening up or down, left or right.

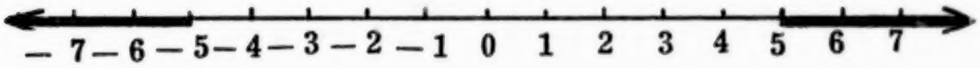
       Verbal problems involving maximum and minimum values.

    4. Illustrate quadratics which are not functions.

*C. Quadratic inequalities

    1. Algebraic solution—example, $X^2 - 25 > 0$ is equivalent to "$X > 5$ or $X < -5$." (See Allendoerfer and Oakley's *Principles of Mathematics.*)

    2. Graphic solution on number line—example, $X^2 - 25 \geq 0$



    3. Absolute value inequalities

       $|x| \leq a$ means $\{x \mid x \leq a\} \cap \{x \mid x \geq -a\}$

       $|x| \geq a$ means $\{x \mid x \geq a\} \cup \{x \mid x \leq -a\}$

IX. Ratio, Proportion, and Variation

A. Definitions

B. Properties of proportions

C. Types of variations (direct, joint, inverse, combined)

D. Verbal problems

*X. Complex Numbers

A. Definitions: equality, addition, multiplication (defined as ordered pairs of reals)

B. Graphical representation of complex numbers

C. Identities and inverses for addition and multiplication

D. The complex number field (all properties satisfied; non-ordered)

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* Topics with asterisks are mainly a part of Algebra II; topics without asterisks are mainly a part of Algebra I.

E. Define—Absolute value, conjugate

F. Redefine complex numbers in form of $a + bi$ (after showing equivalence to (a,b)).

G. Operations with complex numbers in form $a + bi$
addition, multiplication, subtraction, and division

H. Quadratic equations with complex roots

1. Algebraic solution

2. Graphical solution

I. Characteristics of roots (discriminant)

***XI. Solving Systems of Equations (two variables)**

A. One linear—one quadratic

Solved algebraically and graphically

B. Both quadratic—solved algebraically and graphically

C. Verbal problems

***XII. Polynomial Functions**

A. Definition

B. The remainder theorem

C. Synthetic division

D. Factor theorem

E. Descartes' Rule

F. Graphs of cubic and quartic functions

G. Roots of Equations of Higher Degree (optional)—approximations if necessary; considering both rational and/or irrational roots.

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* Topics with asterisks are mainly a part of Algebra II; topics without asterisks are mainly a part of Algebra I.

*XIII. Exponential and Logarithmic Functions

- A. Exponential function ($Y = a^x$, $a > 0$, $a \neq 1$)
  - 1. Define and give domain and range; domain—all real numbers; range—all positive reals
  - 2. Graph—regular, semi-log, and log-log graphs
- B. Logarithmic function ($y = \log_a x$, $a > 0$, $a \neq 1$)
  - 1. Define and give domain and range; domain—set of positive reals; range—all reals
  - 2. Laws of logarithms and proofs
  - 3. Some computation
  - 4. Change of base
- C. Solving exponential and logarithmic equations.

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* Topics with asterisks are mainly a part of Algebra II; topics without asterisks are mainly a part of Algebra I.

CHAPTER 3

Senior High School Geometry

JUNIOR HIGH SCHOOL geometry has been previously discussed in Chapter 1, Fundamentals of Mathematics. This section considers all the geometry taught in senior high school except for the analytic geometry discussed under Advanced Mathematics.

I. A Point of View

A. Objectives of Instruction in Geometry

Principal among the justifications given for including a geometry course in the senior high school curriculum have been the following:

1. *Information:* Geometry contains much information which every educated person needs to know in order to act intelligently in the world of today and tomorrow.
2. *Culture:* The development of geometry reflects in an important way the growth of civilization. It is a fundamental part of our culture; knowledge of geometry contributes to an understanding of our cultural heritage.
3. *A Way of Thinking:* Geometric proof is an instance of deductive reasoning. As a particularly fine model of such reasoning, it facilitates a consideration by high school students of logic. The most important objective is the third—deduction—since the other two are objectives of the junior high school geometry course of study as well. These objectives have been clearly stated in such references as the following, hence they need not be elaborated upon here:

Commission on Mathematics. *Report of the Commission on Mathematics, Program for College Preparatory Mathematics.* Princeton, New Jersey: College Entrance Examination Board, 1959, pp. 22-28.

Butler and Wren. *The Teaching of Secondary Mathematics,* 3rd Edition. New York: McGraw-Hill Book Company, 1960. pp. 475-478.

In teaching geometry, each teacher should strive to satisfy these large and important objectives.

B. Need for Modification of the Traditional Course of Study

Many authorities have recently urged a modification of the traditional geometry course. A principal reason for change is the fact that traditional courses are merely weakened versions of approximately five of the thirteen books of Euclid's *Elements*. As such, these courses are not only pedagogically unsound, but have also perpetuated these logical defects in that 2500-year old text:

a. An incomplete set of axioms

b. Confusingly stated definitions and postulates

c. An unnecessary and logically confusing distinction between postulates and common notions (axioms)

d. A failure to specify a set of undefined terms.[1]

Suggestions by the Commission on Mathematics[2] and experimentation with School Mathematics Study Group [3] and University of Illinois Committee on School Mathematics[4] texts indicate that some of these logical errors can be corrected with a resulting improvement in pedagogy. Other conditions which demand modification of the traditional course:

(1) Solid geometry is not important enough, relative to other mathematics, to deserve a half-year's study separate from plane geometry, and important gains can be secured by teaching the two in the same year.

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[1] Curtis, Charles W., et al. *Studies in Mathematics, Volume II: Euclidean Geometry Based on Ruler and Protractor Axioms,* 2nd revised edition. New Haven, Connecticut: Yale University Press, 1961.

[2] Commission on Mathematics. *Program for College Preparatory Mathematics.* Princeton, New Jersey: College Entrance Examination Board, 1959.

[3] School Mathematics Study Group. *Mathematics in High School: Geometry. Parts I and II.* New Haven, Connecticut: Yale University Press, 1961.

[4] UICSM. *Unit 6: Geometry.* Urbana, Illinois: University of Illinois Press, 1961.

(2) A number of other important geometric topics, such as an introduction to coordinate geometry, receive little or no attention.

(3) The language and operations of set theory can be used to simplify the language and relations of geometry.

Suggestions for modifying the curriculum so as to correct these errors of omission and commission are considered at length in the following section. It is hoped that these recommendations will do at least three things:

(1) Help individual teachers improve their instruction

(2) Help county groups develop new courses of study

(3) Provide a guide for textbook ratings in considering new adoptions.

II. A Modified Course of Study

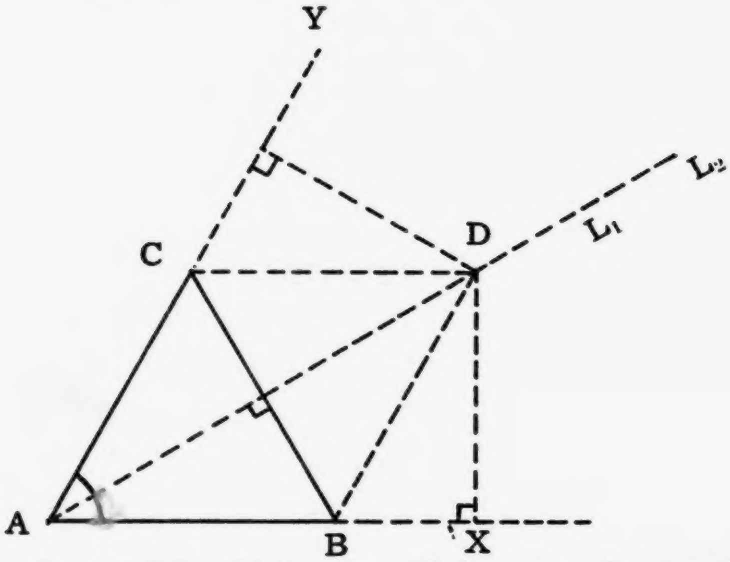
A. Superposition and Betweenness

The principal difficulty with the present course of study is the unnecessary and pedagogically unsound adherence to logical defects in Euclid's *Elements*. For one thing, Euclid used superposition in his congruence proofs without including a superposition postulate. Some high school texts have attempted to correct this situation by adding a superposition postulate. But the result of this is still pedagogically as well as logically unsatisfactory.

This guide strongly recommends that enough of the triangle-congruence statements be postulated so that the remainder may be proven in a way that is logically and pedagogically sound. (See sequence below for suggestions.)

Secondly, Euclid failed to include any postulates of betweenness. This makes it possible to "prove" such invalid statements as:

Every triangle is isosceles.



Consider $\triangle ABC$, with L_1 the angle bisector of $\angle A$ and L_2 the perpendicular bisector of side BC.

There are several possible situations to consider. This sketch of a proof is restricted to the case where L_1 and L_2 are not parallel and intersect exterior to $\triangle ABC$.

Then right triangles ADX and ADY are congruent, and right triangles DBX and DCY are congruent. These imply, respectively, that

$$AX \cong AY, \text{ and}$$

$$BX \cong CY$$

Hence $AB \cong AC$, and the triangle is isosceles.

This difficulty lies in Euclid's failure to give any betweenness postulates, which would have made it possible to specify logically the order of points A, B, and X on line AB, and of A, C, and Y on line AC.

Straightening out this difficulty is not easy, even though it was first accomplished over fifty years ago. To take care of the difficulty within a teachable geometry for high schools is even more troublesome. It is on the basis of settling this betweenness difficulty that the geometry recommendations of the Commission of Mathematics, the Ball State Group, and the School Mathematics Study Group differ the most.

The Commission advocates a recognition of the difficulty, but recommends against adding any betweenness postulates. Under such an arrangement, the teacher would point out to each class where the difficulties lie.

The Ball State Group advocates a set of postulates something like those of Hilbert.

Still a third approach is suggested by SMSG. Following the lead of earlier suggestions by Birkhoff and Beatley,[5] SMSG makes use of the set of real numbers in geometry. A one-to-one correspondence between that set and the set of points on a line, along with the order properties of the real numbers, makes it possible to handle the order of points on a line in terms of the order of the real numbers. Key statements in such a development are the so-called Ruler and Protractor Axioms.

At the time of writing this guide, it is difficult to determine subjectively which of these three suggestions is preferable; experimentation over a period of time would probably be necessary to reach a sound decision. The Curriculum Guide Committee therefore does not now recommend any of the three. Rather it encourages leading teachers and secondary schools in the State of Florida to participate in experimentation with the three approaches. Indeed, perhaps each of these recommendations will find a place in actual instruction according to the whims of individual teachers, schools, and school systems.

B. A Suggested Sequence, with Methodological Recommendations

In any event, certain geometric topics should be a part of any curriculum regardless of the manner of handling the betweenness problem. There follows a sample listing of geometry topics which the Curriculum Guide Committee suggests to Florida's mathematics teachers for inclusion somewhere in the senior high curriculum. These units comprise somewhat more than the usual one-year course.

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[5] Birkhoff, G. D. and Beatley, R. *Basic Geometry*. New York: Chelsea Publishing Company, 1959.

This sequence is divided into thirteen sections as follows:

- I. Nature of a Deductive Geometry
- II. Congruence
- III. Geometric Inequalities
- IV. Parallelism
- V. Similarity
- VI. Coordinate Geometry
- VII. Solid Geometry
- VIII. Locus
- IX. Circles
- X. Area and Volume
- XI. Constructions
- XII. Numerical Trigonometry
- XIII. A Sampling of Other Geometries.

In the following each section is divided into two parts: (1) an outline of topics, and (2) a discussion of the outline in terms of mathematics and pedogogy.

It is not expected that all topics listed in this suggested sequence could be covered in a one-year course in all schools perhaps not even with a selected group of extremely bright youngsters. But certainly the subject matter listed in Sections I through X should be studied, with selections from the remaining three sections (constructions, numerical trigonometry, and other geometries) to be made by individual teachers, school faculties, or county groups as time permits. Another possibility is for the inclusion of such geometry in the later course entitled Fifth Year Mathematics.

Section I—Nature of Deductive Geometry

A. Outline

1. Review and extension of knowledge of common geometric figures and their properties: line, half-line, ray, segment, angle, triangle, etc.
2. Nature of mathematical system
    - a. Undefined terms
    - b. Definitions
    - c. Postulates
    - d. Theorems

3. Nature of proof
   a. Elementary introduction to symbolic logic
   b. Elementary proofs
      (1) Vertical angles are congruent.
      (2) Complements (supplements) of the same angle or congruent angles are congruent.
4. Coincidence properties

B. Discussion of outline

1. Review and extension of knowledge of common geometric figures

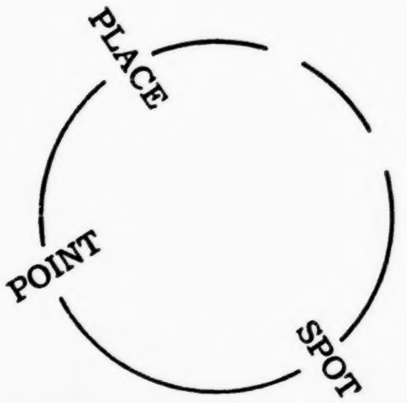
   In line with previous suggestions (see Fundamentals section) and with junior high geometry recommendations by such national groups as SMSG, it is expected that by 1966 students will enter the senior high school geometry course with a much improved and more extensive knowledge of geometry than was previously the case.

   But in any case, a transition will still be necessary from the informal and intuitive geometry of junior high to the logical framework of a senior high geometry course. This transition consists of pulling together all previously learned bits of information. This set of statements and words should then be sorted out and expanded by the teacher into the usually acknowledged subjects of a deductive geometry: undefined terms, definitions, postulates, and theorems. This was the kind of step taken by Euclid which constituted his principal and monumental contribution to knowledge. Statements about which the students are unsure could be classified as conjectures.

2. Nature of Mathematical System

   a. Undefined terms

      Some terms must be left undefined in order to avoid circularity. This can be dramatically illustrated to the class by having individual students trace the meaning of any word via the dictionary—such as follows for the word *point*.



The undefined terms for a senior high school geometry should be *point, line, plane,* and *space.* The universal set will be a plane when doing *plane* geometry, and space while doing solid geometry. Members of the universal set are points. A plane is a special subset of space, and a line is a special subset of a plane; each is given its properties by postulates of the system.

Thus, even though *point, line, plane,* and *space* are undefined, set theoretic relationships exist among them. Any point X is a member of the plane U; $X \in U$. Any line A is a certain collection of points; $A \subset U$.

Even though such words are undefined, a teacher should take time to discuss synonyms for and pictures of these entities so that a student will believe that he understands the ideas involved.

b. Definitions

A definition is a biconditional statement that is logically valid by agreement. The distinction between an undefined word and a defined word may be brought out by a definition of *angle.*

A set of points is an angle if and only if the set is the union of two non-collinear rays whose intersection is the endpoint of each ray.

An undefined word like *line* may be discussed in hand-waving fashion and may be used in axioms, theorems, or conjectures. But a defined term, like *angle,* has a different character. It is a word that replaces a whole set of words; the replacement usually serves the purpose of brevity. (See 24th Yearbook, chapter by Fouch and Nichols, for expansion.)

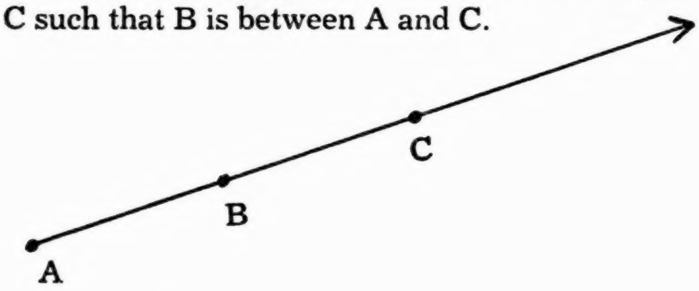
The logical status of a definition is that of a valid biconditional. (See section on logic.) Such a biconditional is valid by *agreement*. In a proof, it may be used in either direction. By stating all definitions in biconditional forms, a teacher may more easily direct a class to use definitions in both directions.

It is strongly suggested that definitions be stated in set theoretic language wherever this achieves greater simplicity and accuracy. Here are some examples:

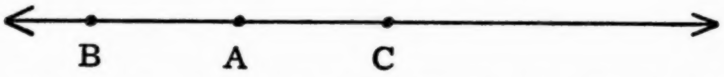
(1) A set of points is a triangle if and only if the set is the union of the three segments determined by three non-collinear points.

(2) A set of points is a line segment AB if and only if the set consists exactly of two points A and B and the points between A and B.

(3) A set of points is a ray $\overrightarrow{AB}$ if and only if the set is the union of line segment $\overrightarrow{AB}$ and the set of points C such that B is between A and C.



(4) $\overrightarrow{AB}$ and $\overrightarrow{AC}$ are opposite rays if and only if A is between B and C.



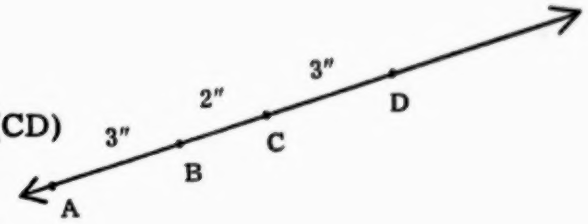
(5) Two angles are vertical angles if and only if their sides form two pairs of opposite rays.

A distinction must be made among:

(a) equality of sets
(b) equality of measures
(c) congruence of sets

Set A is equal to set B if and only if A is a subset of B and B is a subset of A. Thus, in the set of points pictured below, where the inch-measure of each line segment is indicated, the following relations hold:

$BC = CB$
$AB \neq CD$
$m(AB) = m(CD)$
$AB \cong CD$
$BC \cong CB$
$AB \ncong BC$, etc.



(The notation used in this guide does not agree with all 1962 experimental texts. This guide does not demand adherence to any one system of notation, but rather suggests that teachers use notations that keep important and distinct ideas clear in the minds of students.) Thus we have this definition:

> A point M is the midpoint of line segment AC if and only if M is between A and C and $AM \cong MC$ (or $m(AM) = m(MC)$).

There are many words which did not appear in conventional geometry texts before 1960, but which now should be in every teacher's vocabulary. They include:

> half-space, half-plane, half-line, interior or exterior of an angle, side of a point on a line, side of a line in a plane, convex, region, measure.

c. Postulates

Euclidean geometry consists of a certain set of propositions. Some propositions must be assumed in order to avoid circularity, as was true with undefined terms. Those propositions assumed true are called "postulates."

3. Nature of Proof

4. Coincidence Properties

Several experimental texts have postulated such statements as the following:

a. Every line contains at least two points.
b. Every plane contains at least three non-collinear points.
c. Space contains at least four non-coplanar points.
d. Given two points, there is one and only one line containing them.
e. Given three points, there is one and only one plane containing them.
f. If two planes intersect, their intersection is a line.

On the basis of such postulates, theorems such as these can be proven:

Two lines have at most one point in common.

If a line intersects a plane not containing the line, then the intersection is a single point.

Given a line and a point not on the line, there is exactly one plane containing both the line and the point.

Given two lines with a point in common, there is a unique plane containing them.

Such theorems require rather fussy proofs, sometimes by indirect proof, even though the statements are obviously true to the average student. For these reasons, the guide recommends that all such statements be accepted as valid on the basis on intuitive discussion.

Statements about regions of the plane (exterior and interior of an angle, etc.) should be handled in intuitive fashion as well, using the language of sets and the set operations of union and intersection.

Section II—Congruence

A. Outline

1. Definitions and postulates for line segment, congruence,

angle congruence, and triangle congruence; some congruence theorems

a. SAS

b. ASA

c. SSS

d. SAA

2. Other Theorems

Isosceles triangle theorem and its converse; angle bisector, and perpendicular bisector theorem.

B. Discussion of Outline

1. Definitions and Postulates

a. Definitions

A familiar definition of triangle congruence is

Definition 1: Two triangles are congruent if and only if they can be made (or moved) to coincide.

Unless "motion" has been defined, or else accepted as undefined and its properties postulated, definition 1 cannot be allowed in a deductive development of geometry. Many traditional texts use the above definition after only an intuitive introduction to superposition. A satisfactory treatment of rigid motion is available,[6] but it is not recommended for tenth graders.

To avoid the concept of motion, congruent triangles may be defined as triangles whose corresponding parts are congruent. Although both SMSG and Ball State treat triangle congruence in this manner, SMSG uses one-to-one correspondence of vertices while Ball State handles it through renaming of vertices.

SMSG Definition: Two triangles are congruent if and only if there exists a one-to-one correspondence of the vertices of one triangle to those of the other triangle such that corresponding angles and corresponding sides are congruent.

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[6] SMSG Geometry, Part II, Yale Press, 1961, Appendix XIII.

In using the above definition, the vertex correspondence is indicated by the order in which the vertices are listed.

Ball State
Definition: Two triangles are congruent if and only if they can be renamed $\triangle ABC$ and $\triangle A'B'C'$ such that $\angle A \cong \angle A'$, $\angle B \cong \angle B'$, $\angle C \cong \angle C'$, $AB \cong A'B'$, $BC \cong B'C'$, and $CA \cong C'A'$.

Most students will be able to use the Ball State definition without actually recording any reassignment of letter names for vertices.

b. Postulates and theorems of triangle congruence

Experiments with constructions or even sketches will lead students to conjectures concerning the combinations of pairs of $\cong$ sides or $\cong$ angles sufficient for triangle congruence. The statements of geometry are about things that must be imagined rather than about drawings. (Is there a pencil mark that has no width?) Yet it is profitable to make and study pictures and three-dimensional models of geometric figures for these reasons:

(1) Drawings help us to imagine the abstract geometric figures and relations.

(2) Drawings give us clues in discovering geometric properties and relationships.

(3) Drawings provide a quick reference to the letter names assigned to (particular) points for purposes of proof.

These are symbolized by the familiar SAS, ASA, SSS, and SAA.

In classes of below-average ability it is advisable to postulate all four of these triangle congruence propositions. With average and above average students the SAS congruence statement may be assumed and the ASA, SSS, and SAA statements proved without resorting to "moving triangles." Rigorous proofs are available in SMSG, Ball State, UICSM, and other materials. In any event, as noted in the introduction to Senior High Geometry, at least one of these statements should be postulated.

2. Other theorems

Although triangle congruence is useful in proofs throughout the geometry course, these theorems follow immediately:

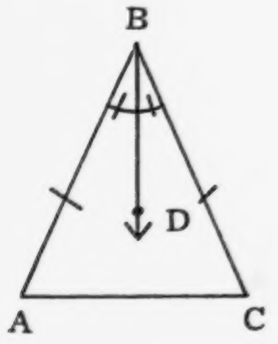
Isosceles triangle theorem: If two sides of a triangle are congruent, the angles opposite them are congruent.

Converse of isosceles triangle theorem: If two angles of a triangle are congruent, the sides opposite them are congruent.

Equilateral triangle theorem: If a triangle is equilateral, then it is equiangular.

Equiangular triangle theorem: If a triangle is equiangular, then it is equilateral.

A proof of the isosceles triangle theorem making use of the bisector of the vertex angle forming congruent triangles should be avoided because of the necessity of proving the bisector $\overrightarrow{BD}$ intersects the base AC between A and C.



Instead, the isosceles triangle theorem may be proved by the congruence, $\triangle ABC = \triangle CBA$. The converse is proved in a similar manner. These proofs are available in SMSG and Ball State materials.

Gifted students could follow this unit with a study of a congruence relation that is applicable to all sets of points. One such treatment defines congruence as an isometric one-to-one correspondence.[7]

Section III—Geometric Inequalities

A. Outline

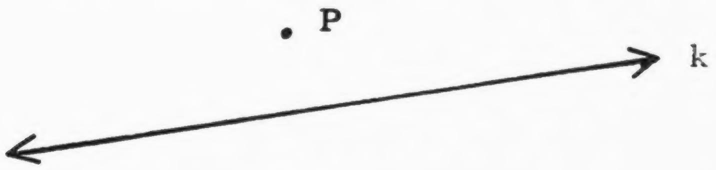
1. If two sides of a triangle are not congruent, the angles opposite these sides are not congruent and the greater angle lies opposite the greater side.
2. If two angles of a triangle are not congruent, the sides opposite these angles are not congruent and the greater side lies opposite the greater angle.
3. An exterior angle of a triangle is greater than either non-adjacent interior angle.

Note: Following this section, a teacher might wish to teach that portion of the Solid Geometry, Section VII, entitled *Perpendicular Lines and Planes in Space.* (See SMSG Geometry.)

Section IV—Parallelism

A. Outline

1. Preliminaries
   a. Definition: In a plane, line m is *parallel* to line k if and only if $m \cap k = \phi$.
   b. Postulate: Through a point P not on a line k, there is exactly one line m such that m is parallel to k.



2. Alternate interior angles theorem, its inverse, and its corollaries

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[7] SMSG, Studies in Mathematics, Volume II, *Euclidean Geometry Based on Ruler and Protractor Axioms,* Chapter 7.

3. Quadrilaterals in a plane: theorems concerning parallelograms, rectangles, squares
4. Angle-sum theorems
5. Optional: elementary discussion of non-Euclidean geometry

   straight-edge geometry

   compass geometry

B. Discussion of Outline

1. This section gives the teacher an opportunity to introduce indirect proof in the proof of the alternate interior angle theorem by means of the exterior-angle-of-a-triangle theorem, and in the proof of the converse by means of the parallel postulate.
2. Optional units on non-Euclidean geometries, straightedge geometry, or compass geometry can provide excellent material for enrichment of bright students at this point in the course.
   a. Non-Euclidean discussion should be concerned with structures that might develop on replacing the 5th Postulate by one or the other counter-postulates. It is important to make clear that these geometries have identically all theorems of Euclidean geometry which result from Euclid's first four postulates. Models must be chosen for simplicity. Extended treatment is not encouraged.
   b. Straightedge geometry (or compass geometry) may be used to stimulate interest and focus attention on the surprising constructional range of these tools used alone.

References: Hudson, Hilda, P. "Ruler and Compass," Part II of *Squaring the Circle*. New York: Chelsea Publishing Company, 1953.

Yates, Robert C. *Geometrical Tools*. St. Louis: Educational Publishing Company, Revised 1949.

Section V—Similarity

A. Outline

1. Preliminaries
   - a. Statements of proportionality
   - b. Definition of similar triangles
   - c. Postulate:
     If a line parallel to one side of a triangle bisects each of the other two sides, then it cuts off proportional segments, and conversely.

2. Theorems

   Theorems of A.A.A., A.S.A., S.S.S., S.A.S.; relationship of similarity to congruence; right triangle theorems leading to Pythagorean Theorem and its converse.

B. Discussion of Outline

Similar triangles are usually defined as two triangles whose corresponding angles are congruent and whose corresponding sides are proportional. Thus a triangle similarity is a special kind of triangle correspondence. From this statement and its converse:

> If a line parallel to one side of a triangle intersects the other two sides, then it cuts off segments which are proportional.

These similar triangle statements follow logically:

(1) If the angles of one triangle are congruent to the corresponding angles of another, the triangles are similar (A.A.A.).

(2) If the sides of one triangle are proportional to the sides of another triangle, the triangles are similar (S.S.S.).

(3) If two sides of one triangle are proportional to two sides of another triangle and the angles included between the pairs of sides are congruent, the triangles are similar (S.A.S.).

We can now develop the similarity statements of the right triangle:

(1) The altitude to the hypotenuse is the geometric mean of the segments into which it separates the hypotenuse.

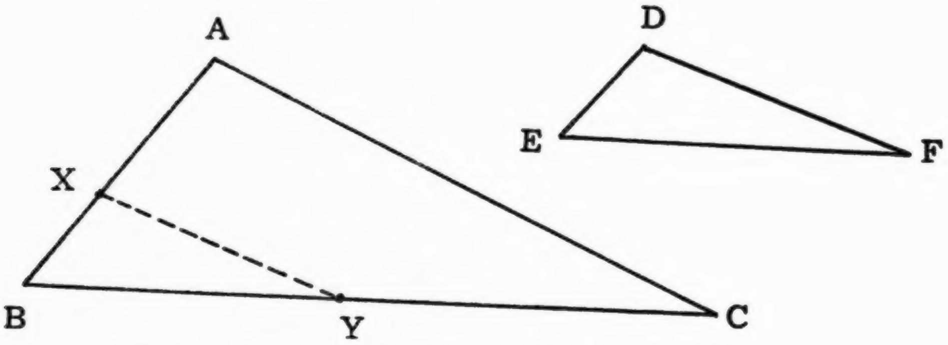
(2) Either leg is the geometric mean of the hypotenuse and the segment of the hypotenuse adjacent to the leg.

(3) The square of the length of the hypotenuse is equal to the sum of the squares of the lengths of the legs—Pythagorean Theorem—and conversely.

$$a^2 + b^2 = c^2$$

It is instructive to compare the different approaches to similarity taken by the Commission on Mathematics, SMSG, and the Ball State Group.

The Commission on Mathematics used the approach implied by the outline of topics above. Following the postulate given there, the Commission suggested the proof of the A.A.A. theorem sketched here:

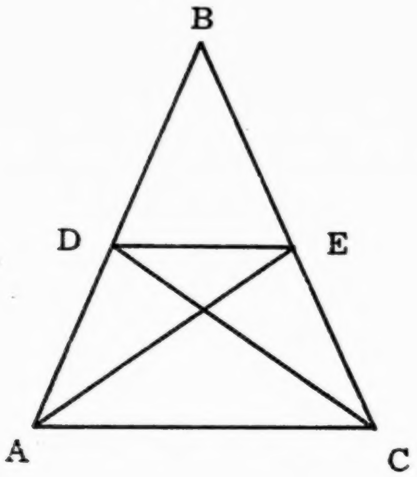


Given: $\angle A \cong \angle D$, $\angle B \cong \angle E$, $\angle C \cong \angle F$.

BX and BY constructed congruent to ED and EF, respectively. Hence $\triangle BXY \cong \triangle DEF$, $\angle BXY \cong \angle D \cong \angle A$, and $\angle BYX \cong \angle F \cong \angle C$. Hence $XY \parallel DF$. Thus, by the postulate, sides are proportional, etc.

The SMSG presents areas of polygons before similarity and proves the Pythagorean Theorem by showing that the area of the square upon the hypotenuse equals the sum of the area of the squares upon the legs.

Because of this, SMSG is able to prove the statement given in the outline above as a postulate, as sketched below:



$\overleftrightarrow{DE}$ is given parallel to $\overleftrightarrow{AC}$

$$\frac{\text{Area } \triangle ADE}{\text{Area } \triangle BDE} = \frac{AD}{BD}$$

$$\frac{\text{Area } \triangle CED}{\text{Area } \triangle BDE} = \frac{CE}{BE}$$

But area $\triangle ADE =$ area $\triangle CDE$

so $\frac{AD}{BD} = \frac{CE}{BE}$ etc.

The Ball State Group gives a proof of the AAA Theorem only in terms of commensurable segments, a disadvantage which is avoided by the Commission and the SMSG.

Section VI—Coordinate Geometry

As soon as the Pythagorean Theorem has been proven, it is possible to do a considerable amount of coordinate geometry. This subject has never been a part of the standard geometry course. In algebra, it has been treated altogether too lightly and in a way which emphasizes graphical algebra rather than any geometric principles. Yet it contains techniques and ideas which are fundamental to a student's success in the calculus; a modest introduction would be extremely useful to the student.

The Guide therefore recommends this sequence:

1. One-to-one correspondence between points of a plane and ordered pairs of real numbers
2. Slope of a line segment; division of line segment by a point into given ratio; midpoint formula; distance formula
3. Slope of a line; perpendicular and parallel lines
4. Equations of a straight line; equation of a circle.

With these key facts, one may prove many theorems of synthetic geometry by analytic means, which is the essence of coordinate geometry. An example is the following:

> The medians of a triangle are concurrent in a point that divides each of the medians in the ratio 2 : 1.

A listing of theorems easily proved by analytic means, along with some sample proofs, is given in the *Appendices to the Report of the Commission.* The number of the theorems to be so done is left to the judgment of school mathematics departments or county groups.

Section VII—Solid Geometry

It is suggested that a separate course in solid geometry be abolished. However, this is not intended to indicate that solid geometry is unimportant. For one thing, much can be gained by teaching plane and solid geometry together: not only does it help the student by having him think at the same time of geometric entities in both two and three dimensions, but also it prevents the student from becoming bored as he sees the heavy-handed proofs of intuitively obvious statements that are so typical of solid geometry. Also, many mathematical applications (*e.g.*, map projection) are very easy to consider in the context of plane and solid geometry together. It is well to stress these applications as well as skill in drawing two-dimensional figures of three-dimensional objects.

The relationship between plane and solid geometry is illustrated in these analogous statements:

Plane Geometry: The universe is a plane.	Definition:	Two *lines* are parallel if and only if their intersection is the empty set.
	Postulate:	Through any point P, not on a given *line* L, there is one and only one *line* parallel to L.
	Theorem:	If two lines are each parallel to a third line, they are parallel to each other.

Solid Geometry: The universe is space.	Definition:	Two *planes* are parallel if and only if their intersection is the empty set.
	Postulate:	Through any point P, outside a given plane M there is one and only one *plane* parallel to M.
	Theorem:	If two planes are each parallel to a third plane, they are parallel to each other.

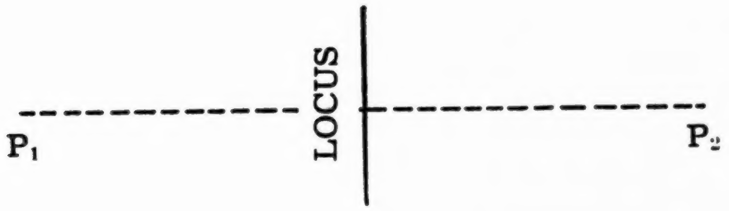
There are certain solid geometry topics, however, which should be taught. A listing of these topics can be found in the Report of the Commission on Mathematics, 1959.

At least three sequences of the solid geometry are possible:

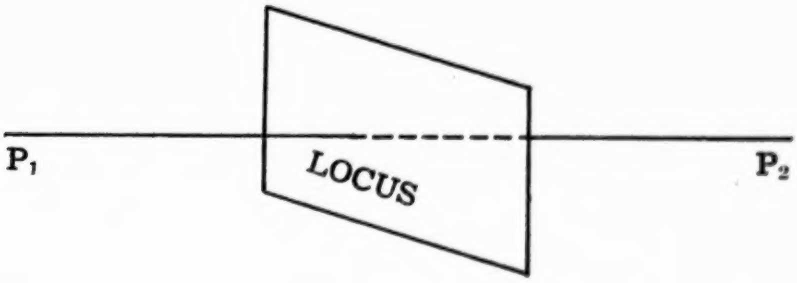
1. Include the solid geometry topics as a separate unit after completing the corresponding plane geometry unit.
2. Extend each two-dimensional discussion to three dimension continuously.
3. Develop the solid geometry at the completion of plane geometry.

An example of the first approach is the following, which also illustrates that confusion may result when attention is given strictly to only plane geometry. Unless the corresponding solid theorem is developed, this theorem is misleading.

Plane: The set of points equidistant from two points is a line which is the perpendicular bisector of the line segment joining the two points.



Solid: The set of points equidistant from two points is a plane which is the perpendicular bisector of the line segment joining the two points.



Section VIII—Locus

Students often misunderstand locus problems in geometry because the wording does not clearly convey to them the actual meaning intended. The language of sets is particularly useful in expressing locus problems.

Consider set A consisting of the numbers 1, 2, 3, 4, and 5. We denote this $A = \{1, 2, 3, 4, 5\}$. Let set $B = \{1, 2, 3, 4, 5, 6\}$. Now set $A =$ set B if and only if each member of set A is in set B and each member of set B is in set A. For the examples given $A \neq B$; although each number of set A is also in set B, 6 belongs to set B and not to set A.

If set $C =$ the set of non-negative integers less than two, and set $D =$ the set of integers n such that $n = n^2$. To show that set $C = D$, two things must be proved.

1. The non-negative integers less than two each satisfy $n = n^2$.
2. Each integer that satisfies $n = n^2$ is also a non-negative and less than zero.

If the phrase "The locus of" is replaced by "the set of all" in the locus theorems of geometry, it becomes clear that most of them require a proof that some set A equals a set B. By way of example, one locus theorem common to most geometry courses is

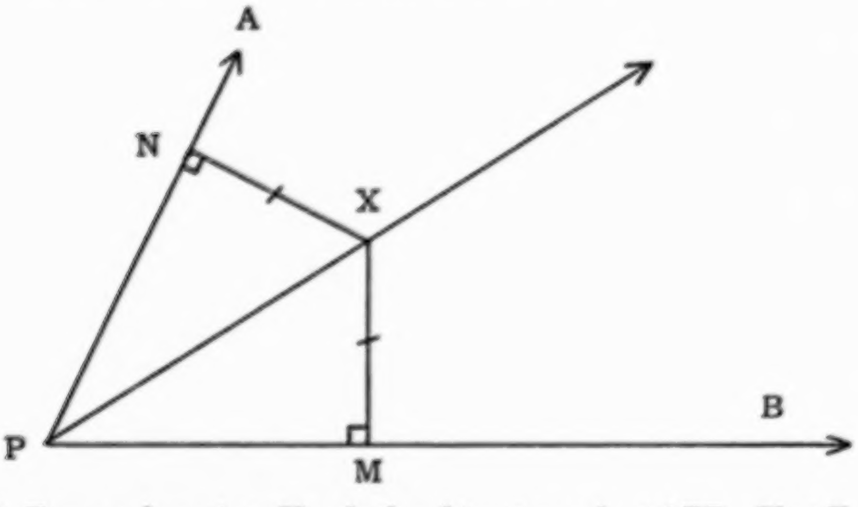
> The locus of all points equidistant from the sides of an angle is the bisector of the angle.

Making the suggested translation in terms of sets of points, the theorem becomes:

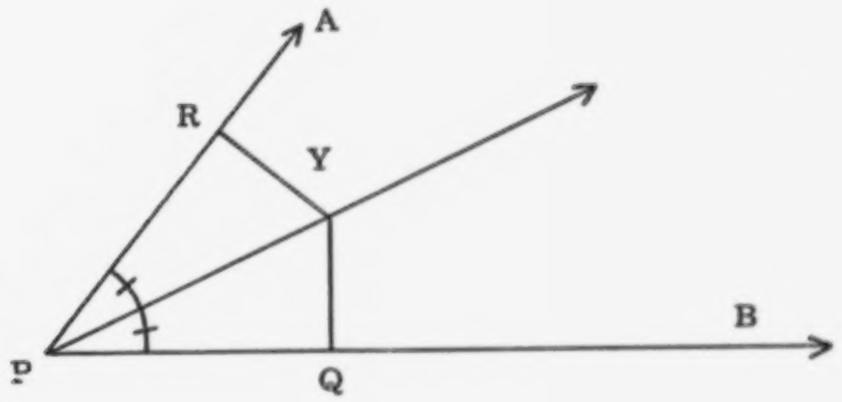
> The set of all points equidistant from the sides of an angle is the bisector of the angle.

Let set $E =$ the set of all points equidistant from the sides of an angle $\angle APB$. Let set $F =$ the bisector of $\angle APB$. To prove this locus theorem, it is necessary to prove set $E =$ set F. As in the previous examples, it is not sufficient to show merely that each point of set E is also in set F. In addition it must be proved that each point belonging to set F is also a member of set E. Thus the familiar two-part proof is necessary.

1. For each point X that is equidistant from the sides of $\angle APB$, XM and XN, the perpendicular distances to sides $\overrightarrow{PA}$ and $\overrightarrow{PB}$ are equal. If $XM \neq P$, $\triangle XNP \cong \triangle XMP$ by the hypotenuse-leg theorem and $\angle NPX \cong \angle MPX$. Hence $\overrightarrow{PX}$ bisects $\angle APB$. If the distance from X to each side is zero then $X = P$ and is also on the bisector of $\angle APB$. Therefore each point of set E is in set F.



2. For each point Y of the bisector of $\angle APB$, $Y \neq P$, $\angle APY \cong \angle YPB$. Let segment YP and segment YQ be the perpendicular segments from Y to $\overrightarrow{PA}$ and $\overrightarrow{PB}$. Then $\triangle PRY \cong \triangle PQY$ by the hypotenuse-acute angle theorem and $YR = YQ$. Hence Y is equidistant from the sides. If $Y = P$, $YR = \text{zero} = YQ$. Therefore each point of set F is in set E.



There is another wording used in locus problems that sometimes confuses students. For example, textbooks sometimes speak of "the locus of a point *moving* in a plane such that the distance between it and a given point is 6 units." Applications such as these probably had their origin in the study of motion in physics. The terminology is unfortunately in conflict with the mathematicians' insistence that points do not move. A more acceptable translation of the above statement is, "The set of all locations in a plane that are 6 units from a given point." It is often pedagogically helpful to imagine a speck actually moving from one to another of these locations. The locations are then the points.

In determining the locus of points satisfying two or more conditions, the language of sets is very useful. If A is the set of points satisfying condition X, and B is the set of points satisfying condition Y, then the set of points or locus of points satisfying both X and Y is $A \cap B$.

The locus unit is a natural place to bring in some coordinate geometry. A point of the cartesian plane may be represented by an ordered real number pair (x,y). A set of points may be characterized by an equation in x and y. The locus of points (ordered pairs) satisfying *two* conditions (equations) is the intersection of the two solution sets.

It may be that students experience more difficulty with the locus terminology than with the concept. SMSG's treatment entirely omits the word "locus" and calls the unit "Characterization of Sets."

Section IX—Circle

NOTE: This topic provides an excellent opportunity to blend solid and plane geometry. For this reason, the two are dealt with side by side. Those who prefer teaching a separate unit in solid geometry should ignore the three-dimensional references.

A. Outline

1. *Definitions*

a. The following definitions are illustrative:
A set of points is a *circle* if and only if the set consists of all points in a plane at a given distance from a specified point in the plane.

A set of points is a *sphere* if and only if the set consists of all points in space at a given distance from a specified point.

A line segment is a *radius* of a circle (sphere) if and only if one end point of the segment is the center of the circle (sphere) and the other end point is on the circle (sphere).

b. Definitions will also need to be given for those standard terms, among others: chord, secant, tangent lines and planes, diameter, tangent segment and ray, secant ray, interior of a circle or a sphere, congruent circles, arc, central and inscribed angles, tangent circles.

2. Important Theorems

a. Tangency and perpendicularity
If a line is perpendicular to a radius at the end point where the radius intersects the circle, then the line is a tangent to the circle.

If a line is tangent to a circle, then the line is perpendicular to the radius at the point of tangency.

(Analogous theorems in solid geometry may be stated by replacing *line* by *plane*, and *circle* by *sphere* in the above.)

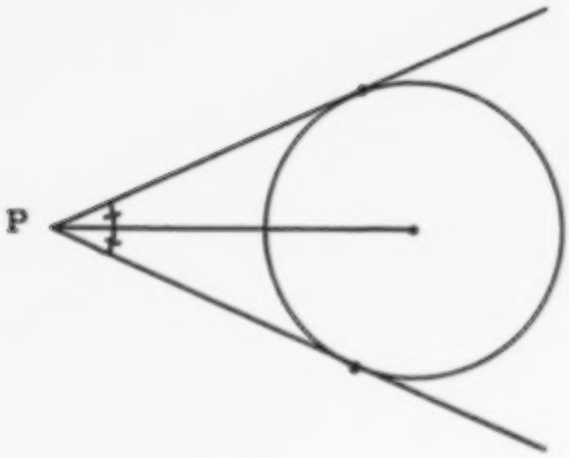
If a plane intersects the interior of a sphere, then the intersection of the plane and sphere is a circle.

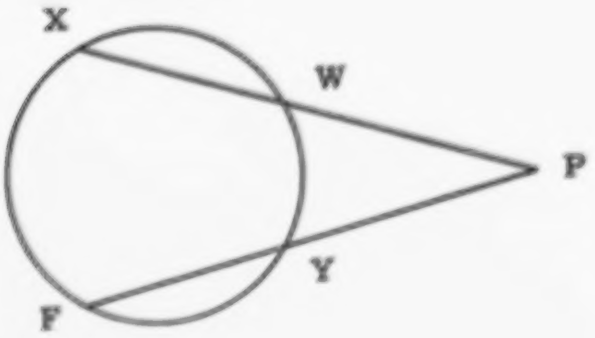
b. Angles, chords, tangents, secants, and intercepted arcs

The measure of an inscribed angle is one-half the measure of its intercepted arc.

If an angle has its center on a circle with one side a tangent ray of the circle and the other a secant ray, then the measure of the angle is one-half the measure of the intercepted arc.

The two tangent segments to a circle from a point P of the exterior are congruent. The segment from the center of the circle to point P bisects the angle determined by the tangent segments.





If P is a point exterior to the circle, W and X are intersections of the circle with a secant ray from P, Y and F are intersections of the circle with a second secant ray, then PW : PX = PY : PF.

c. Circles in a coordinate plane
The graph of the equation

$$(x - a)^2 + (y - b)^2 = r^2$$

is the circle with center at (a.b) and radius of length r.

Section X—Area (Optional)

Many prefer to include a deductive treatment of area in a geometry course, even though students may be familiar with the formulas for the area of standard polygonal regions.

Such a treatment is reasonably standard, even in a contemporary course. Teachers should be careful to distinguish here, however, between a triangle and a triangular region, a rectangle and a rectangular region—in general, between a polygon and a polygonal region. A polygonal region is the union of the polygon and its region. One should be aware that we determine the area of a polygonal region rather than of a polygon—although we may still use a language elision to speak of the *area of a polygon* for the sake of convenience.

(Special note: If area is definitely to be covered in a course, one may wish to teach this topic before similarity. By doing so, the basic proportionality theorem:

> If a line is parallel to a side of a triangle and intersects the other two sides, then it cuts off segments which are proportional to these sides.

may be proven by using this theorem concerning area of triangular regions:

> If two triangles have equal altitudes, then the ratio of their areas is equal to the ratio of their bases in the same order.

For further detail, see *Geometry, Part II* by the School Mathematics Study Group, and the section above concerning similarity.)

Section XI—Constructions (Optional)

Constructions play an important role in the study of geometry, and particularly is this true in the plane. They not only stimulate interest but focus attention on basic principles and proven facts. The first four postulates of Euclid in effect give us the line of indefinite length on two points, or the circle through a given point with center at a second given point. For physical pictorial models, the unmarked straightedge and the compasses used only in the manner prescribed by the postulates produce a structure marvelous in its elaboration and extent.

Purists, somewhat rightfully, object to the use of the marked straightedge (ruler), the carpenter's square, the protractor, the dividers, etc., as instruments permissible in a theoretical geometry course. They should be allowed only with full realization of their "non-Euclidean" character.

It is recommended that problems of construction appear early in the course and continue throughout. Certainly they would seem to be particularly essential in any unit on locus, and in the unit on parallelism (see above).

Section XII—Numerical Trigonometry (Optional)

When time is available, many teachers like to include in a geometry course a short unit in trigonometry. It has made the two contributions of serving:

1. as an application of the concept of similar triangles;
2. as a preview to the trigonometry course to follow.

The first contribution still exists, but a danger is now present in the second in that the course of study in trigonometry has radically changed to emphasize its function and analytic properties rather than the numerical. (See the trigonometry section of this guide.)

Should a teacher still wish to teach a short unit in the usual numerical trigonometry, care should be taken to warn the students that they are covering a particular subset of trigonometry that is quite different from and less important than what they will study in a later course.

Section XIII—Other Geometries (Optional)

The Commission on Mathematics and other national authorities have recommended the teaching of geometries other than

Euclidean—whenever there is time. Units I-X above constitute a full course in a normal situation; Units XI and XII constitute enrichment. Further enrichment may be obtained by teaching elements of such other geometries as topology, projective geometry, and vector geometry.

Satisfactory brief treatments of topology can be obtained by using sections of such books as the following:

Arnold, B. H. *Intuitive Concepts in Elementary Topology.* Englewood Cliffs, New Jersey: Prentice Hall, 1962.

Johnson, D. A. and Glenn, W. H. *Topology: The Rubber Sheet Geometry.* St. Louis: Webster, 1960.

A satisfactory brief treatment of projective geometry may be obtained by selecting from portions of almost any projective geometry book.

The concept of vector is becoming more and more important every day in higher mathematics. Teachers are well advised to look into such approaches to vector geometry as is to be found in the 23rd Yearbook of the National Council of Teachers of Mathematics, Chapter VI, by Walter Prenowitz. Also crucial is a more algebraic treatment of vectors, as mentioned in the advanced mathematics section of this guide.

CHAPTER 4

Advanced High School Mathematics

MANY HIGH-SCHOOL students take a fourth year of mathematics following the two years of algebra and one year of geometry previously outlined. A few honor students in some secondary schools may take an additional fifth year of mathematics.

Such a fourth and fifth year of mathematics will be referred to in this guide as Advanced High School Mathematics.

The first of these two courses in advanced high-school mathematics will mainly consist of trigonometry, analytic geometry, and advanced topics in high-school algebra. These may be taught as an integrated course or as separate courses. All students completing these portions of advanced high-school mathematics in one year should be able to enroll in a beginning course in calculus upon entrance to college. All other students who are not able to complete this full outline should expect to take a pre-calculus course after entering college.

The second of the two courses in advanced high-school mathematics will be referred to in this guide as Fifth Year Mathematics. Such a course is for the few honor students who probably start algebra in the eighth grade and complete at least the first course of Advanced High-School Mathematics in the junior year.

A short but important unit in symbolic logic that may be covered in two to three weeks is considered desirable in addition to other topics. It would cover such things as implication, conjunction, disjunction, valid and invalid arguments, converse, inverse, contrapositive, direct and indirect proof, counter-example and possibly quantifiers. This may be done informally or may be enhanced with truth tables at the discretion of the teacher. This might be the first topic and be used throughout other topics or come later.

Trigonometry

Trigonometry had its origin in early historical times. Hipparchus (180 BC-125 BC), a leading astronomer, first scientifically formulated the subject to assist in the calculation of the positions of heavenly bodies. Ptolemy (AD 150) popularized Hipparchus' theories, following which there were few important additions to trigonometric content until the seventeenth century.

Hipparchus and Ptolemy based their efforts on the theory of similar triangles—today there are other and more generally applicable mathematical theories upon which trigonometry may be developed. The motivations of the ancient mathematicians were in the applications of trigonometry to related efforts in astronomy, navigation, and surveying—today there are more challenging and timely applications evident in science and technology. This is especially true in statics and dynamics, electromagnetic waves, and vibration problems of all kinds. All this calls for a radical revision of the trigonometry course of study.

The following outline for trigonometry has been organized to meet contemporary needs, as recommended in the Report of the Commission on Mathematics of the College Entrance Examination Board.

Trigonometry may be offered as a one-semester course or may be integrated with the other mathematics courses described in this guide under Advanced Mathematics. In either event, all topics listed in this outline should be covered.

There is an alternative to the following recommended sequence of topics: it is possible to develop the essentials of trigonometry by starting with the circular functions (as in III) in terms of real numbers and following with angles, radian measure, and degree measure. In the words of the Report of the Commission on Mathematics:

> Such passage from "pure" to "applied" trigonometry may have greater brevity and mathematical elegance, but it presents a higher level of abstraction to the learner.[1]

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[1] *Report of the Commission on Mathematics, Program for College Preparatory Mathematics.* Princeton, New Jersey: College Entrance Examination Board, 1959 p. 19.

Outline for Trigonometry

I. Background for the Study of Trigonometry

In the "Background for the Study of Trigonometry" it is necessary that the student become familiar with the properties of the field of real numbers and their geometrical representation to establish the rectangular coordinate system and the function concept. The amount of time spent on this unit will depend on the background of the student. For some it would be a simple review, whereas for others it will be necessary to logically establish the coordinate plane which is the basis for the definitions of the trigonometric functions as well as the graphical representation of these functions and their inverses.

A. The Real Number System

  1. Properties of the Real Number System

    a. Inequalities—Order Relations

    b. Absolute Value

B. A Rectangular Coordinate System

C. The Distance Between Two Points

D. The Function Concept

E. The Graph of a Function

II. The Trigonometric Functions—Coordinate Approach

The Commission on Mathematics recommends that the trigonometry of grade eleven be centered on coordinates, vectors, and complex numbers.[3] It is expected that the student of trigonometry is familiar with the coordinate plane and the concept of the one-to-one correspondence between the set of points in a plane and the set of ordered pairs of real numbers. This understanding was developed in previous courses and reviewed in Unit I of this outline.

A. Angles

  1. Definition

An angle may be defined as the union of two rays. In trigonometry, however, it is desirable to define an angle as being formed by one ray rotating about a fixed point on a fixed ray. The fixed point is called the vertex, the fixed ray is called the initial side, and the rotating ray is called the terminal side of the angle.

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[3] *Report of the Commission on Mathematics, Appendices*, p. 186.

2. Standard Position—any quadrant

   An angle is in standard position on the coordinate plane if its vertex is at the origin and its initial side is on the positive x-axis.

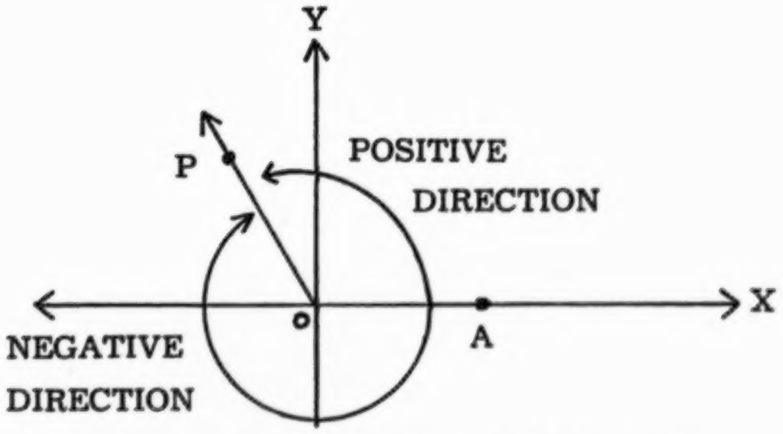


Figure 1. Angle AOP is in standard position.

3. Coterminal angles

   Coterminal angles are angles in standard position which have the same terminal side. Since the amount and direction of rotation is not limited, infinitely many angles have the same terminal side as a given angle in standard position. The difference in the number of rotations of any two angles in a set of coterminal angles is an integer.

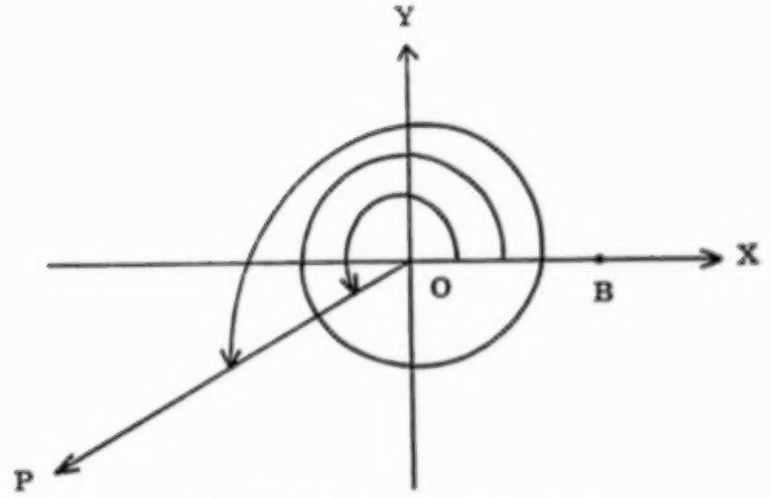


Figure 2. Coterminal Angles

4. Measure of an angle

Any measure of the amount of rotation of the terminal side of an angle in standard position is called the *measure* of the angle. The two basic units of measure of an angle are the degree and the radian.

One rotation is equal to 360°. The measure of an angle is the product of the number of rotations and 360°. A degree is divided into smaller units called the minute and second. One minute is 1/60 of a degree and one second is 1/60 of a minute.

A radian is the measure of a central angle of a circle which determines an arc of the circle that is equal in length to the radius of the circle.

These two units, degree and radian, are easily related. By the above definitions, one rotation is equal to 360° and to 2π radians. Hence π radians $= 180°$.

B. The Six Trigonometric Functions—definition in terms of coordinate plane and angle in standard position.

The trigonometric ratios are defined in the following manner: If θ is an angle in standard position on the coordinate plane, and if P is a point on the terminal side of θ being r units from the origin and having coordinates (x,y) then:

		Abbreviation
sine $\theta = \frac{y}{r}$		sin θ
cosine $\theta = \frac{x}{r}$		cos θ
tangent $\theta = \frac{y}{x}$	$(x \neq 0)$	tan θ
cotangent $\theta = \frac{x}{y}$	$(y \neq 0)$	cot θ
secant $\theta = \frac{r}{x}$	$(x \neq 0)$	sec θ
cosecant $\theta = \frac{r}{y}$	$(y \neq 0)$	csc θ

These definitions have no meaning when the denominators equal zero.

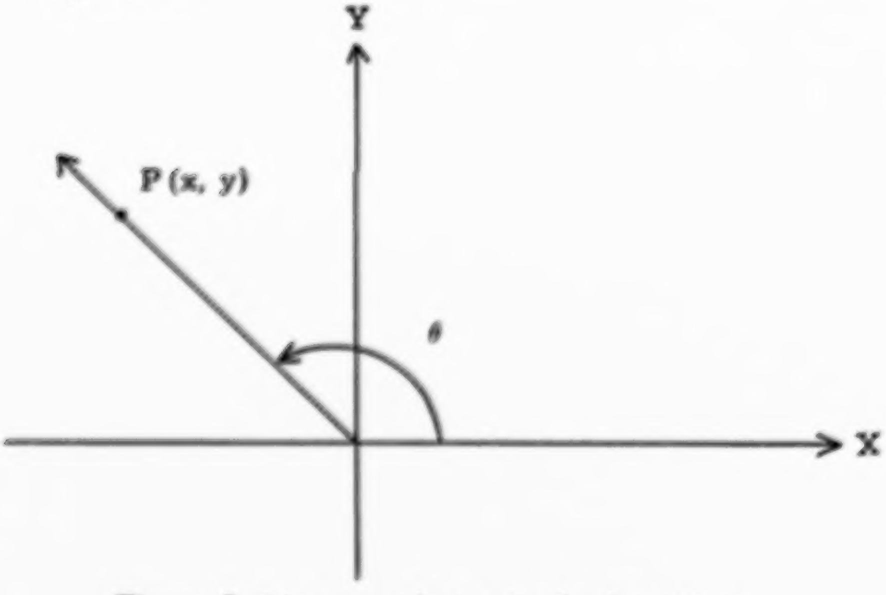


Figure 3. θ is an angle in standard position.

C. Reciprocal Relations of Functions—derived from definition

D. Ratio Relations of Functions

tangent $\theta = \frac{\text{sine } \theta}{\text{cosine } \theta}$ and cotangent $\theta = \frac{\text{cosine } \theta}{\text{sine } \theta}$ as derived from definitions.

E. Signs of the Values of the Trigonometric Functions—established from definitions and coordinate plane

F. Trigonometric Values for Special Angles

1. Quadrantal Angles
2. 30° - 45° - 60° Angles
3. Coterminal Angles
4. Complementary Angles proved geometrically

G. Pythagorean Relationships

H. The Values of All Functions Derived From the Value of One

III. Circular Functions

A. Definition of Circular Functions

In order to properly define the circular functions and to distinguish them from the trigonometric functions, it is necessary to establish a correspondence between the real numbers and the points on a unit circle.

A circle of radius one unit and center at the origin on the coordinate plane is given. Using the same unit of measure and a counter-clockwise direction as positive, a Euclidean number line is wrapped around the unit circle, letting 0 of the number line coincide with the point (1,0) of the plane. Then for each real number there is a unique point on the unit circle. See Report of the Commission on Mathematics, Appendices, pp. 206-9.

Two definitions are necessary: (1) coincident numbers: numbers represented by the same point on the unit circle, (2) primary numbers: x is primary number of $0 \leq x < 2\pi$.

Definition of Circular Functions.

If t is a real number represented by the point P (x,y) in the unit circle, then $\sin t = y$, $\cos t = x$, $\sec t = \frac{1}{x}$, $\tan t = \frac{y}{x}$, $\csc t = \frac{1}{y}$, $\cot t = \frac{x}{y}$, if such a ratio is defined.

B. An Isomorphism Between the Circular and the Trigonometric Functions

An isomorphism between the circular functions and the trigonometric functions exists.

To establish an isomorphism between these two sets it is necessary that we show a one-to-one correspondence between the sets which is preserved under addition and multiplication.

We first establish a one-to-one correspondence between the set of all real numbers and the set of all angles in the following manner: If an angle is in standard position on the plane (coordinate plane), the terminal side will intersect the unit circle in exactly one point. Hence with each primary angle in standard position there is associated exactly one primary real number. If the point representing a primary real number is connected to the origin, exactly one primary angle is formed. Hence with each primary real number there is associated exactly one primary angle in standard position.

If t_1 is a primary real number associated with the primary angle θ_1 and t_2 is a primary real number associated with the primary angle θ_2 then $(t_1 + t_2) = t_3$ is a real number which can easily be shown to be associated with the angle $(\theta_1 + \theta_2) = \theta_3$. Also if t_1 is a primary real number associated with the primary angle θ_1 and if n is any real number, it can likewise be shown that nt_1 will be the real number associated with angle $n\theta_1$.

These two correspondences can be established by referring to the theorem from plane geometry which states that a central angle of a circle is measured by the arc which it determines.

A correspondence of definitions follows on the next page:

TRIGONOMETRIC FUNCTIONS	*CIRCULAR FUNCTIONS*
angle in standard position	real number on the unit circle
primary angle	primary number
coterminal angles	coincident numbers
reference angle	reference number
$\sin \theta$	$\sin t$
domain of $\sin \theta$ (all angles)	domain of $\sin t$ (all real numbers)
range of $\sin \theta$, $(-1) \leq \sin \theta \leq 1$	range of $\sin t$, $(-1) \leq \sin t \leq 1$
$\cos \theta$	$\cos t$
domain of $\cos \theta$, all angles	domain of $\cos t$ (all real numbers)
range of $\cos \theta$, $(-1) \leq \cos \theta \leq 1$	range of $\cos t$, $(-1) \leq \cos t \leq 1$
$\tan \theta$	$\tan t$
domain of $\tan \theta$ (all angles except $90° + n \cdot 180°$) where n is an integer	domain of $\tan t$ [all real numbers except $(\frac{\pi}{2} + n\pi)$] where n is an integer
range of $\tan \theta$ (all real numbers)	range of $\tan t$ (all real numbers)

Because the two sets are isomorphic, any theorem established in one is valid in the other with the proper correspondence.

C. Radian Measure to Relate Circular and Trigonometric Functions

Since there exists a one-to-one correspondence between all real numbers and all angles it is sometimes desirable to use

the number corresponding to an angle as the measure of the angle. This scheme provides us with another close tie between trigonometric and circular functions. The units of measure of an angle in this manner is called a radian and is established by the following definition:

> If a central angle of a circle determined an arc which is equal in length to the radius of the circle, the measure of the angle is one radian.

It is easy to establish that the real number on the unit circle which corresponds to a primary angle is the measure of the angle in radians.

IV. Periodicity and Graphs of Trigonometric Functions

A. Graphs of the Trigonometric Functions

B. Graphs of y = A sin Bx, y = A cos Bx

C. Graphs of Composite Functions

V. Trigonometric Identities

Identities should not be treated as equations. A valid proof of an identity should consist of changing one side of the identity into the other by a series of proper substitutions of the fundamental identities or by changing both sides to the same expression in the same manner.

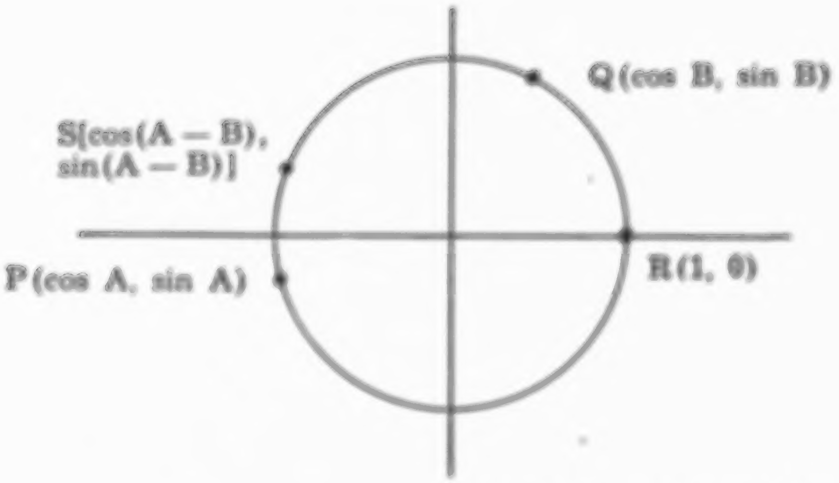
A maximum of five class periods should be spent on identities since the students will get further practice with them in working with equations and multiple angle formulas.

VI. Trigonometric Equations

VII. Multiple Angle Formulas

A. Addition Formulas (using distance formula)

Let P be a point on the unit circle representing the real number A and let Q be a point representing the real number B. Then the coordinates of P are (cos A, sin A) and the coordinates of Q are (cos B, sin B)



Then there exists another point S on the unit circle representing the real number (A — B) and the coordinates of S are [cos (A–B), sin (A–B)].

Let R be the point (1,0).

Since the length of arc RQ is the real number B and the length of arc SP is also the real number B, then arc RS = arc QP. Each is obtained by adding B to the length of arc SQ.

Since equal arcs have equal chords in the same circle, chord RS = chord QP.

by the distance formula:

Chord $QP = \sqrt{(\cos B - \cos A)^2 + (\sin B - \sin A)^2}$

and chord $RS = \sqrt{[1 - \cos (A-B)]^2 + [0 - \sin (A-B)]^2}$

equating and solving for cos (A–B), we obtain:

$\text{Cos } (A-B) = \cos A \cos B + \sin A \sin B$

Other multiple angle formulas are established by proper substitutions in the above formula.

See Cameron, E. A. *Algebra and Trigonometry.* New York: Holt, Rinehart and Winston, 1960.

B. Double-angle Formulas

C. Half-angle Formulas

D. Product Formulas

VIII. Inverse Trigonometric Relations

A. Graphing of Inverse Functions and Relations

B. Solution of Equations Involving Inverse Trigonometric Relations[3]

IX. Solution of Triangles

This section of the outline deals with applications of trigonometry. Special tables, computing machines, and other equipment have made the logarithmic solution of triangles an almost obsolete tool. However, seven or eight lessons devoted to the solution of triangles will lay the foundation for any future work in triangle solution and vectors.

A. The Right Triangle

B. The Oblique Triangle

The solution of oblique triangles should be limited to the use of the Laws of Sines, Cosines, and Tangents. Many other formulas could be used to make computation easier, but with the advent of the calculator there is no need to avoid the arithmetic involved.

1. Law of Sines
2. Law of Cosines
3. Law of Tangents

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[3] The inverses of the trigonometric functions are not functions unless the domain is restricted. It is suggested that the inverse relations be graphed first, then the necessary restrictions be placed on the domain to obtain a function.

X. The Complex Number System

The complex number system includes all other number systems such as integers, rationals, reals which have been studied. It completes the development of our numbers. It is a field as are the rationals and the reals, but it has a property not possessed by these sub-fields, namely, it is algebraically closed.

A. Definition and Properties of Complex Numbers

This section is a review of complex numbers as developed in algebra. For a careful development of complex numbers the reader is referred to the algebra outline in this bulletin. For further discussion reference may be made to:

1. Chapter Eleven, "Algebra and Trigonometry" by Edward A. Cameron. Holt, Rinehart and Winston, 1960.
2. Chapter Five of "Integrated Algebra and Trigonometry," Fisher and Ziebur. Prentice Hall, 1958.

B. Graphical Representation

1. The Complex Plane

A complex number is determined by a pair of real numbers. This immediately suggests a method of representing complex numbers as points in a plane. When this is done the plane is referred to as the complex plane.

2. Vectors

The idea of vector is basic to physical science and of great importance in the development of contemporary higher mathematics. For further discussion of vectors, reference should be made to the Report of the Commission on Mathematics, Appendices, Chapter 19 and Chapter 20.

C. Trigonometric Form—Polar Coordinates

D. Operations on Complex Numbers

E. De Moivre's Theorem

F. Roots of a Complex Number

Appendix of Additional Topics

I. Measurement

- A. Approximation
- B. Significant Figures
- C. Accuracy

II. Scientific Notation and Logarithms (before study of triangles in Section IX)

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Advanced Algebra

Some of the topics listed here may be included in an earlier algebra course for especially able students. Conversely, some strengthening may be needed on topics studied in previous algebra courses. Special attention should be given to polynomial, logarithmic, and exponential functions.

I. Sequence and Series

- A. Define sequence and series
- B. Examples and notation (use sigma notation as well as listing first terms)
- C. Finite and infinite series
- D. Arithmetic series
  - 1. Definition
  - 2. Nth term, sum, and arithmetic mean

E. Geometric series

   1. Definitions

   2. Nth term, sum, geometric mean, and repeating decimals

F. Harmonic series (definition and examples)

G. Verbal problems

II. Mathematical Induction

III. Permutations, Combinations, Probability

A. Permutation

   1. Definition

   2. Problems

   3. Factorial notation, define 1! and 0!

B. Combinations

   1. Definition

   2. Problems

C. Binomial theorem

   1. Pascal's triangle

   2. Binomial coefficients

   3. General term

   4. Proof of theorem (optional)

D. Probability

   1. Definitions

   2. Use of combinations and permutations

   3. Probability of an event; independent events

   4. Conditional probability

IV. Determinants, Vectors, and Matrices

A. Determinants

   1. Definitions and notation

2. Using determinants to solve simultaneous linear equations
3. Define—minor, cofactor
4. Expansion by minors
5. Properties of determinants

B. Vectors
1. Definitions and notation
2. Addition of vectors
3. Scalar
4. Multiplication (inner product or dot product)
5. Length of a vector

C. Matrices
1. Definition and notation
2. Order of dimension
3. Equality, addition, multiplication (by a scalar; by a matrix). Point out multiplication not commutative.
4. Inverse of square matrix
5. Application to simultaneous equations

V. Introduction to Modern Algebra (Optional)

A. Properties of groups and fields with examples from rational numbers, real numbers, and modulo arithmetic (prime and non-prime). Also examples such as

$$\{x \mid x = a + b\sqrt{3}\}$$

B. Careful deduction of theorems

C. The abstract field (Keeping "+" and "•" as operations, but observing abstract nature of consequences of the axioms.)

Analytic Geometry

An analytic geometry course may appropriately be offered the second semester of the fourth year. It would be a course with such flexibility that the teacher could adapt the content of the course to

1. complement previous mathematics courses
2. provide a means of introducing concepts that are normally found in college mathematics courses
3. provide an opportunity for original work on the part of the student.

The content, however, should contain at least the following topics:

1. The coordinate system
2. Graphs and equations of lines
3. Discussion of the graph of an equation
4. Circles
5. Conic sections

These topics would constitute a minimum program on which the teacher would base an appropriate course for his students.

Analytic Geometry Outline

I. Basic Concepts

- A. Coordinate systems
  1. One-dimensional

     (Point out one-to-one correspondence between the set of points on a line and the set of real numbers.)

  2. Two-dimensional

     (Point out one-to-one correspondence between the set of points in a plane and the set of ordered pairs of real numbers.)

  3. Three-dimensional

     (Point out one-to-one correspondence between the set of points in space and the set of ordered triples of real numbers.)

- B. Distances
  1. Directed distances
  2. Undirected distances

    3. Distance formula
    4. Division of a line segment into a given ratio.

II. The Straight Line

- A. Slope and inclination
    - 1. Parallel lines and perpendicular lines
    - 2. Angle of inclination
    - 3. Angle between two intersecting lines

(Two cases in which the formula cannot be used:

If two lines are perpendicular to each other.

If one of the lines is parallel to the Y-axis.

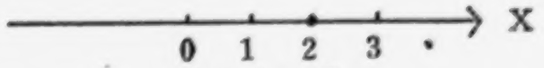
In the first case the formula is not needed anyway and in the second case it can be shown that $\tan \phi = \frac{1}{\tan \theta_1}$ if L_2 is parallel to the Y axis and $\tan \phi = -\frac{1}{\tan \theta_2}$ if L_1 is parallel to the Y axis.)

- B. Equation of a line
    - 1. Definition: An equation of a line L is an equation in x and y with the following properties:
        - a. If point $P \in L$, then the coordinates (x,y) of P satisfy the equation.
        - b. If $P \notin L$, then (x,y) does not satisfy the equation.
    - 2. Point-slope form
    - 3. Slope-intercept form
    - 4. Intercept form
    - 5. Show that every straight line in the coordinate plane is an equation of the first degree in x and y and that the graph of any equation of the first degree in two variables, such as $Ax + By + C = 0$ is a straight line.

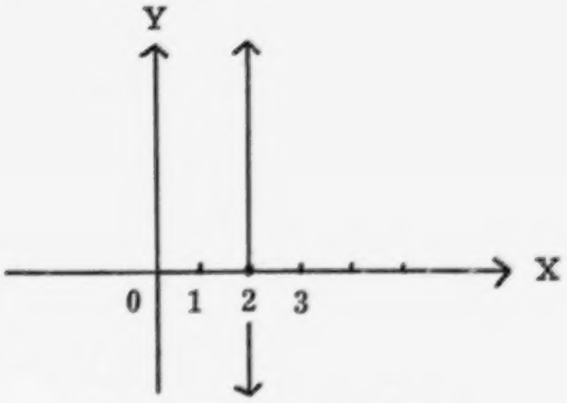
6. Note the difference between the graphs of the sets

$S_1 = \{x \mid x = 2\}$ $\qquad$ $S_2 = \{(x,y) \mid x = 2\}$

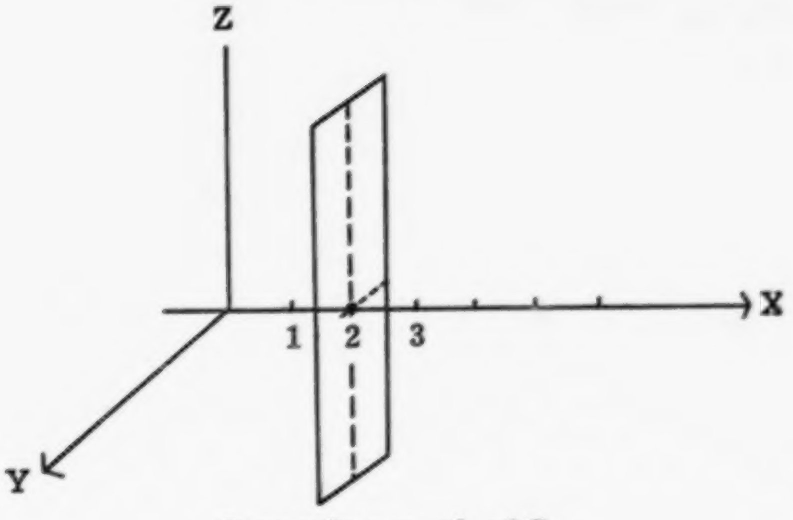
$S_3 = \{(x,y,z) \mid x = 2\}$



Graph of S_1



Incomplete graph of S_2



Incomplete graph of S_3

7. Linear inequalities in two variables

(Show that the point (x_1, y_1) is on, below, or above the line with equation $y = mx + b$ depending on whether $y_1 = mx_1 + b$, $y_1 < mx_1 + b$, or $y_1 > mx_1 + b$.)

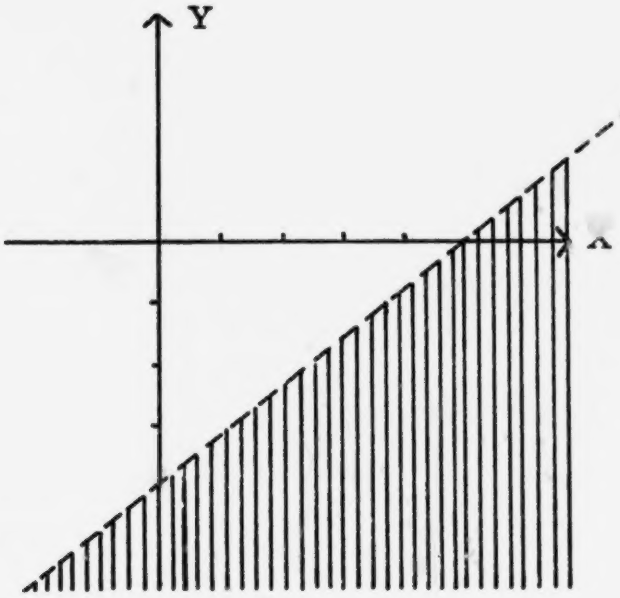
3

Example:

$S = \{ (x,y) \mid 4x - 5y > 20\}$

$$y < \frac{4}{5}x - 4$$

Hence the graph of S consists of all points which lie below the line L with equation $y = \frac{4}{5}x - 4$



Incomplete graph of S

8. Distance from a line to a point

C. Families of Lines

D. Locus derivation problems

Definition: The equation of a locus S is one with the following properties:

If $P \epsilon S$, then (x,y) satisfiies the equation;

If (x,y) satisfies the equation, then $P \epsilon S$.

III. Conic sections

A. Introduction

1. Method of obtaining conics

2. General equation of a conic

   $Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$

B. Circle

1. Definition: a circle is the set of all points in a plane that are equidistant from a fixed point of the plane.

2. Derive the center-radius form of an equation of a circle

   $(x - h)^2 + (y - k)^2 = r^2$

   where (h,k) is the center and r is the radius.

3. Show that $x^2 + y^2 + Dx + Ey + F = 0$ is the general form of an equation of a circle.

   To graph this equation, write the equation in the center-radius form

   $(x + \frac{D}{2})^2 + (y + \frac{E}{2})^2 = N$

   where $N = \frac{1}{4}(D^2 + E^2 - 4F)$.

   If $N > 0$, then the graph is a circle with center

   $\left(-\frac{D}{2}, -\frac{E}{2}\right)$

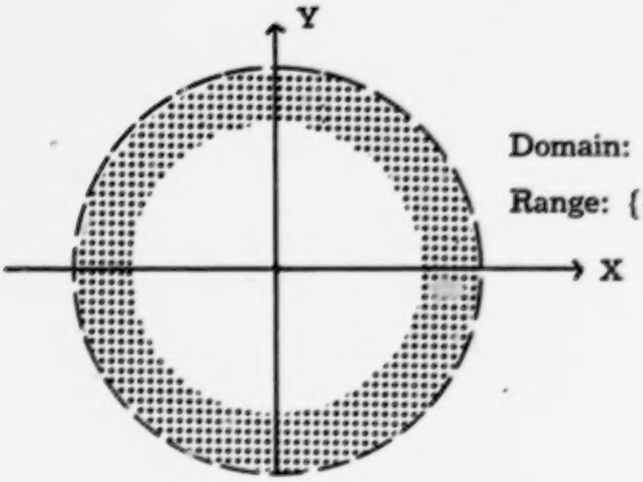
   and radius of $\sqrt{N}$.

   If $N = 0$, then the graph is the single point $\left(-\frac{D}{2}, -\frac{E}{2}\right)$

   If $N < 0$, then the graph is the null set.

4. Graph of some relations

a. $\{(x,y) \mid x^2 + y^2 \geq 9 \text{ and } x^2 + y^2 < 16\}$

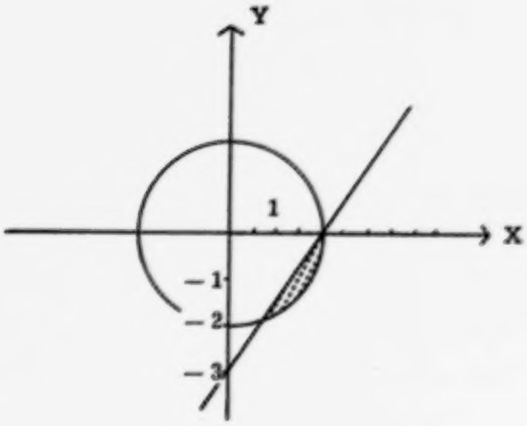


Domain: $\{-4 < x < 4\}$

Range: $\{-4 < y < 4\}$

The graph of the relation is the shaded portion.

b. $\{(x,y) \mid x^2 + y^2 \leq 4 \text{ and } 3x - 2y \geq 6\}$



Domain: $\{\frac{10}{13} \leq x \leq 2\}$

Range: $\{-\frac{24}{13} \leq y \leq 0\}$

(Determine the domain and range by finding the intersection points of the curves.)

5. Families of circles

C. Parabola

1. Definition: A parabola is a set of points in a plane so that for each point the undirected distances from a fixed point and from a fixed line are equal.

2. Derive the equations of a parabola

   Show the effects of the constants on the curve.

3. Discuss vertex, axis, symmetry, focus, and directrix.

4. Graph some relations such as:

   a. $R = \{ (x,y) \mid y^2 = 2x \text{ or } y = x - 4\}$

   b. $R = \{ (x,y) \mid y^2 < -9x \text{ and } 3x^2 = 8y\}$

   Give the domain and range of each of the relations.

5. Applications

   Path of a projectile, headlights, arches of some bridges, radar screens, cable of a suspension bridge, some lawn sprinklers.

D. Ellipse

1. Definition: An ellipse is the set of points located in a plane so that the sum of the undirected distance of each point from two fixed points is a constant.

   Also define focal axis, vertices, major axis, minor axis and center.

2. Derive the equation of an ellipse with center at (h,k).

   Discuss the eccentricity.

3. Graph of some relations

   (Similar to the ones mentioned above.)

4. General equation of ellipse

   $Ax^2 + Cy^2 + Dx + Ey + F = 0$ where $AC > 0$ and $A \neq C$.

The graph may be determined as follows:

$$A(x + \frac{D}{2A})^2 + C(y + \frac{E}{2C})^2 = N$$

a. If $N = 0$, then the graph is a single point.

b. If $N > 0$, then the graph is an ellipse with center at

$$\left(-\frac{D}{2A}, -\frac{E}{2C}\right)$$

c. If $N < 0$, then the graph is the null set.

5. Applications

Orbits of planets about the sun, springs used in some cars, elliptic gears.

E. Hyperbola

1. Definition: A hyperbola is the set of points located in a plane so that the difference of the undirected distances from two fixed points is a constant.

Also define foci, focal axis, center, latus rectum, asymptotes, tranverse axis.

2. Equation of a hyperbola

Show the effects of the constants

Equation of an equilateral hyperbola

Discuss the eccentricity

3. Graph some relations

(Similar to ones mentioned above)

4. General equation

$$Ax^2 + Cy^2 + Dx + Ey + F = 0$$

where $AC < 0$

The graph may be determined as follows:

$$A(x + \frac{D}{2A})^2 + C(y + \frac{E}{2C}) = N$$

a. If $N = 0$, then the graph is two intersecting lines.

b. If $N \neq 0$, then the graph is a hyperbola with the center at $\left(-\frac{D}{2A}, -\frac{E}{2C}\right)$

5. Applications

For locating the place from which a sound, such as gunfire or radio beams, emanates or for tracking space objects by radio or radar. Same principle is also used by radar-equipped airplanes to determine their locations.

F. General second degree equation

1. Simplification by translation of the axes
2. Simplification by rotation of the axes
3. Evaluation of the invariant $B^2 - 4AC$ to determine the curve.

IV. Rapid sketching of *simple* curves

A. Algebraic curves

1. Discuss intercepts, symmetry, asymptotes, tangents at the origin, and excluded values.
2. Sketch polynomial functions.
3. Sketch rational functions.
4. Sketch other algebraic functions, such as $y^2 = \frac{x-1}{x^2-4}$.
5. Sketch by composition of ordinates.

B. Transcendental curves

1. Logarithmic curve
2. Exponential curve
3. Trigonometric curves
4. Combinations of these

V. Polar coordinates

A. Polar coordinate system

B. Rapid sketching of simple curves in polar form

  1. Point plotting

  2. Determine symmetry, intercepts, and excluded values of θ

C. Polar form of straight line and conics

D. Point out it is possible to get exactly the same graph for two non-equivalent equations.

  Example: $r = 5 \cos (\frac{\theta}{2})$ and $r = -5 \sin (\frac{\theta}{2})$

  Same graph, but different solution sets.

VI. Parametric equations

A. Parametric representation of straight line and conics

B. Plotting a curve by a table of values

C. Elimination of parameter

VII. Space coordinates

If time permits the inclusion of this topic, see the development in *Calculus with Analytic Geometry,* by Johnson and Kiokemeister, Allyn & Bacon (state-adopted text).

Fifth-Year Mathematics

It is generally agreed that courses should be designed for the accelerated student who has completed the equivalent of the previous material suggested in this guide by the end of his eleventh year, and is interested in an additional year of mathematics. It should be especially noted that there should ordinarily be only a small percentage of students state-wide who would have attained a sufficient mastery of mathematics to have a real need for such a course.

If the need is apparent, then the following offerings are recommended:

1. If topics in number theory and such modern algebra as Matrix Algebra have not been included and adequately covered previously, then a combination of one-semester courses in these two areas are suggested. Currently, the School Mathe-

matics Study Group materials would serve to outline the content in Matrix Algebra and there are several texts available in number theory which are adequate.

2. For those whose interests may lie in the biological or physical sciences, as well as in mathematics, a one-semester course in probability and statistics might well be substituted for one of the previously suggested courses.

3. For some students, particularly those who are accelerated in the physical sciences and plan to further their study in this area, as well as in mathematics, there is a need for better preparation in calculus. The Physical Science Study Committee physics course and the trend toward offering two years of physics and chemistry in high school are contributing factors in creating this need. It is strongly recommended, however, if a course involving the fundamentals of calculus is to be offered, that it be a full-year course in calculus and be taught as outlined by the Advanced Placement Program under the auspices of the College Entrance Examination Board so that the members of the class will be able to qualify for advanced standing or credit in college mathematics.[4]

The above suggestions by no means exhaust the possible offerings for the academically able student. These offerings might also include topics from computer mathematics, Boolian Algebra, symbolic logic, and linear programing. In addition, some of the optional units from the senior high geometry course would serve nicely here: construction theory, vector geometry, projective geometry. In the first case, the impossibility of certain constructions (*e.g.*, angle trisection) may form a highly satisfactory several-week unit, now that the students are proficient in algebra and trigonometry.

It should be kept in mind that, if the students have achieved a sufficient mastery of mathematics to be ready for the advanced courses, the courses should be of sufficient depth to insure proper understanding of the fundamental concepts. In considering which of these courses to use, primary consideration will probably have to be given to the background and training of available teachers.

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[4] *Advanced Placement Program Syllabus*. College Entrance Examination Board. c/o Educational Testing Service, Princeton, New Jersey.